Center for Scientific Computation
And Mathematical Modeling

University of Maryland, College Park

L^1 Techniques for various Problems in:

Nonlinear PDEs, Numerics, Signal and Image Processing

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The Inviscid, 2D Nonlinear Transport:

A BV Approach*

* with Dongming Wei (Maryland)

2D nonlinear transport: from viscous to the inviscid

- The 1D setup — $u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon$ (Burgers eq.):

1. Compactness: $\|u^\epsilon(\cdot,t)\|_{BV} \le \|u_0^\epsilon\|_{BV} \Longrightarrow u^\epsilon \to \overline{u}$ ("L^1 hard");

2. Conservation form: $u_t^\epsilon + \frac{1}{2}\Big((u^\epsilon)^2\Big)_x = \epsilon u_{xx}^\epsilon$ (obvious);

3. Passing to the limit: $\overline{u}_t + \frac{1}{2}(\overline{u}^2)_x = 0$ ("diagonal" regularity);

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1. We prove compactness: $\|\mathbf{u}^\epsilon(\cdot,t)\|_{BV} \lesssim \|\mathbf{u}_0^\epsilon\|_{C_0^1}$ ("L^1 hard");

2. Conservation form is not obvious: $\mathbf{u}_t^\epsilon + \nabla_{\mathbf{x}}(\mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon) \neq \epsilon \Delta \mathbf{u}^\epsilon$;

3. Limit $\mathbf{u}^\epsilon \to \overline{\mathbf{u}}$ exists — but what is the dynamics of $\overline{\mathbf{u}}(\cdot,t)$:

$$\text{“}\overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \overline{\mathbf{u}} = 0\text{”?} \qquad \text{(Zeldovitch system)}$$

BV bound by spectral dynamics

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \nabla_{\mathbf{x}} \mathbf{F}, \quad \mathbf{F} = \mathbf{F}[\mathbf{u}, \partial_{\mathbf{x}_i} \mathbf{u}, \partial^2_{\mathbf{x}_i, \mathbf{x}_j} \mathbf{u}, \dots]$$

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- viscosity forcing: $\mathbf{F}[\dots] = \epsilon\Delta\mathbf{u}$

$\odot$ If $M = \nabla\mathbf{u}$ is symmetric with $\lambda_1(\mathbf{x}, t) < \dots < \lambda_N(\mathbf{x}, t)$:

$$\begin{aligned} \partial_t\lambda_1 \;+\; \mathbf{u}\cdot\nabla_{\mathbf{x}}\lambda_1 + \lambda_1^2 &\geq \epsilon\Delta\lambda_1 \\ \partial_t\lambda_N + \mathbf{u}\cdot\nabla_{\mathbf{x}}\lambda_N + \lambda_N^2 &\leq \epsilon\Delta\lambda_N \end{aligned}$$

- Ricatti equation $\Longrightarrow \lambda_1 < \dots < \lambda_N \leq Const. \mapsto \nabla_{\mathbf{x}}\mathbf{u} \leq Const.$

Uniform BV bounds on viscous transport eqs

- A bound on the spectral gap: $\eta := \lambda_N - \lambda_1$

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#2. 2D nonsymmetric M: $\quad L^1$ spectral dynamics of symm$M$

Uniform BV bound on $\mathbf{u}^\epsilon$ of transport eq.: $(\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \epsilon \Delta \mathbf{u}$

Passage to the limit: $\mathbf{u}_t^\epsilon + \mathbf{u}^\epsilon \cdot \nabla_{\mathbf{x}} \mathbf{u}^\epsilon = \epsilon \Delta_x \mathbf{u}^\epsilon$

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- Dual solution: the viscous case

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The inviscid case: L^∞-ϕ exists by the OSLC $\nabla_{\mathbf{x}} \cdot \mathbf{u} \le Const.$

$$\phi_t + \nabla_{\mathbf{x}}(\mathbf{u}\phi) = 0, \quad \phi_T(\cdot) \mapsto_{\mathbf{u}} \phi(\cdot,t), \ t \in [0,T];$$

$\mathbf{u}$ is a 'dual solution' if for <u>all</u> $\phi_T(\cdot)$: $\Big(\phi_T, \mathbf{u}(\cdot,T)\Big) = \Big(\phi(\cdot,0), \mathbf{u}_0\Big)$

L^1 Convergence Rate Estimates

for Hamilton-Jacobi Eqs*

* with Chi-Tien Lin

Convex HJ eqs: $\varphi_t^\epsilon + H(\nabla_{\mathbf{x}}\varphi^\epsilon) \approx 0$

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HJ eqs with convex Hamiltonian: $D^2_{\mathbf{p}}H(\mathbf{p}) > 0$

Example: eikonal eq.: $H(\varphi) = \frac{1}{2}|p|^2$

Viscosity slns, $\varphi_t + H(\nabla_{\mathbf{x}}\varphi) = 0$, demi-concave: $D^2\varphi(\cdot, t) \leq k(t)$

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$\Longrightarrow$ L^1 ERROR ESTIMATE: $supp\, \varphi^\epsilon = \Omega_0 \subset \mathrm{IR}^N$

$$\|\varphi^\epsilon(\cdot, t) - \varphi(\cdot, t)\|_{L^1} \lesssim_t \overbrace{\|\varphi_0^\epsilon - \varphi_0\|}^{\text{initial error}}_{L^1(\Omega_0)} + \overbrace{\|\varphi_t^\epsilon + H(\nabla\varphi^\epsilon)\|}^{L^1 \text{ consistency}}_{L^1([0,t]\times\Omega_t)}$$

Examples of approximate viscosity solutions

- Convex Hamiltonians — L^1-framework: $\varphi_t^\epsilon + H(\nabla\varphi^\epsilon) = \epsilon\Delta\varphi^\epsilon$

$$\|\epsilon\Delta\varphi^\epsilon\|_{L^1([0,T]\times\Omega_t)} \lesssim \epsilon \implies \|\varphi(\cdot,t) - \varphi^\epsilon(\cdot,t)\|_{L^1} \lesssim_t \|\varphi_0^\epsilon - \varphi_0\|_{L^1} + \epsilon$$

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  Crandall-Lions, Souganidis, Oberman, ...

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L^1-rate + demi-concavity (interpolation): L^∞-rate of order $\sim \sqrt{\epsilon}$

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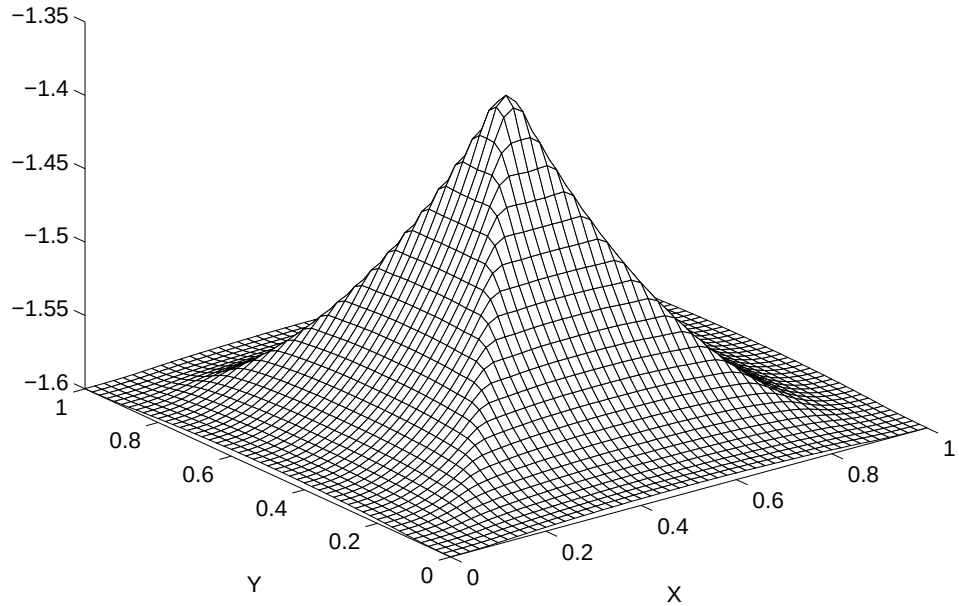
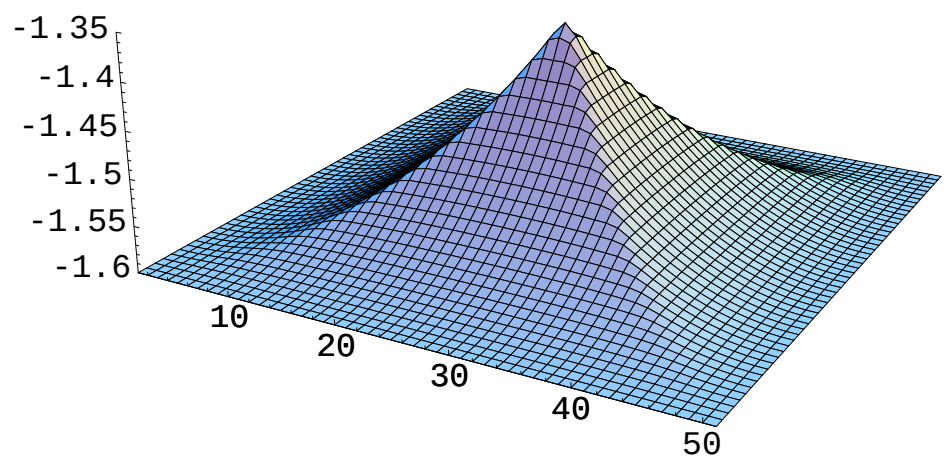
- Godunov type schemes: $\varphi^{\Delta\mathrm{x}}(\cdot,t^{n+1}) = P_{\Delta\mathrm{x}}E(t^n+\Delta t,t^n)\varphi^{\Delta\mathrm{x}}(\cdot,t^n)$

$$\boxed{\|\varphi^{\Delta\mathrm{x}}(\cdot,t) - \varphi(\cdot,t)\|_{L^1} \lesssim_t \frac{t}{\Delta t}\Big\|\big(I - P_{\Delta\mathrm{x}}\big)\varphi^{\Delta\mathrm{x}}(\cdot,t)\Big\|_{L^1([0,t]\times\Omega_t)}}$$

1st- and 2nd-order demi-concave projections: $\|I - P_{\Delta\mathrm{x}}\|_{L^1} \sim |\Delta\mathrm{x}|^2$

Beyond monotone schemes for HJ eqs





Central-scheme solution of eikonal eq.
with second-order limiters (Lin-T. and Kurganov-T.)

Edge Detection in piecewise smooth spectral data by Separation of Scales: A BV approach*

*with S. Engelberg (Jerusalem) and J. Zou (Wachovia)

Gibbs phenomenon and concentration factors

$$S_N[f](x) = \sum_{-N}^{N} \widehat{f}(k) e^{ikx}, \quad \widehat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy$$

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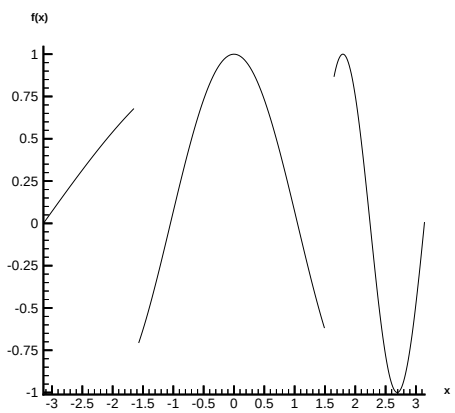

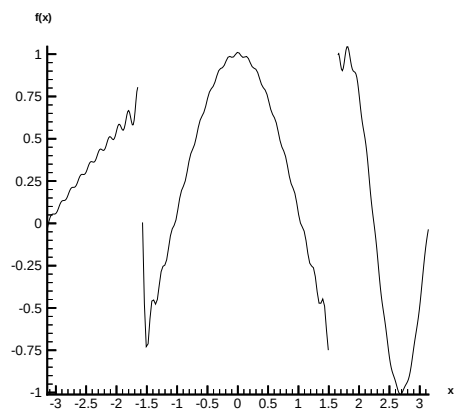

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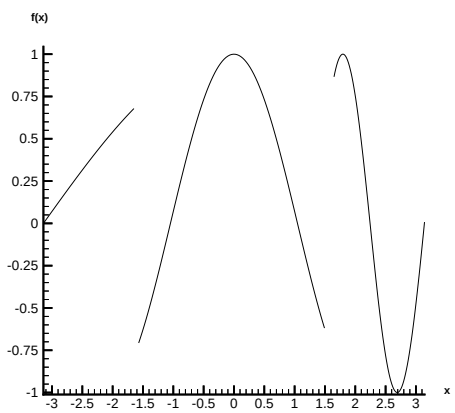

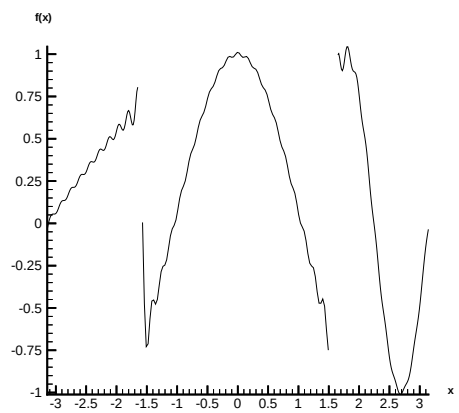

- First-order accuracy (global): $\quad |\widehat{f}(k)| \sim [f](c) \dfrac{e^{-ikc}}{2\pi i k} + \mathcal{O}\left(\dfrac{1}{|k|^2}\right)$

Gibbs phenomenon and concentration factors

$$S_N[f](x) = \sum_{-N}^{N} \widehat{f}(k) e^{ikx}, \qquad \widehat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy$$

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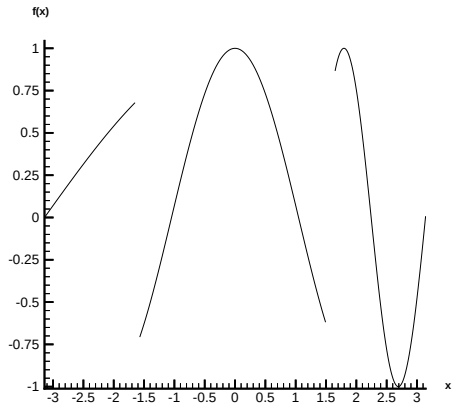

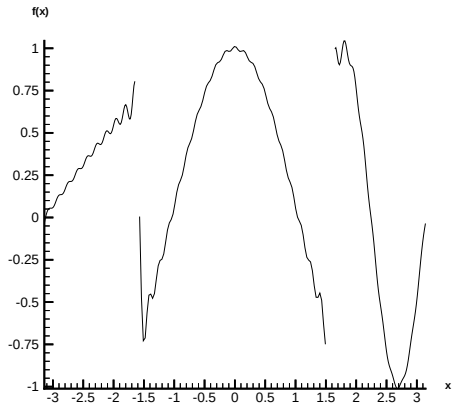

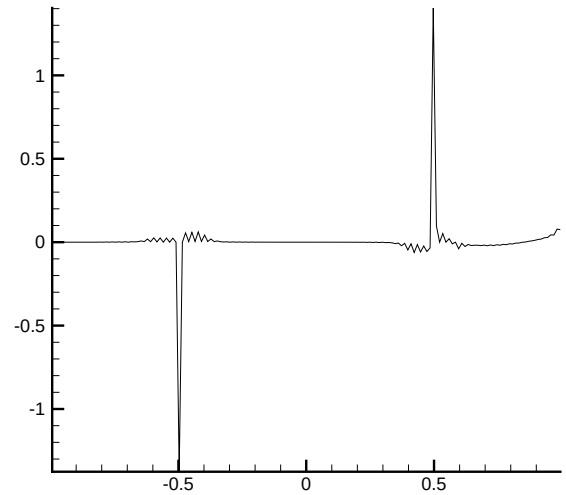

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$$\boxed{S_N^\sigma[f] = [f](x) + \mathcal{O}(\varepsilon_N) \sim \begin{cases} \mathcal{O}(1), & x \sim singsupp[f] \\ \mathcal{O}(\varepsilon_N), & f(\cdot)_{|\sim x} \text{ smooth} \end{cases}}$$

Edge detection by separation of scales: $\varepsilon_N \sim \dfrac{1}{(N \times dist(x))^{s_\sigma}} \ll 1$

$$\sum \frac{\pi i}{c_\sigma} sgn(k)\sigma\left(\frac{|k|}{N}\right)\hat{f}(k)e^{ikx} -$$ how to choose $\sigma(\theta_k)$?

$$S_N^\sigma[f] = K_N^\sigma * S_N f \equiv K_N^\sigma * f:$$

$$\boxed{K_N^\sigma(x) := -\frac{1}{c_\sigma}\sum_{|k|\le N}\sigma\left(\frac{|k|}{N}\right)\sin kx}$$

- 'Heroic' cancelation of oscillations — <u>all</u> $\sigma's$ are admissible:

$$K_N^\sigma(x) \sim \frac{\cos(N+\frac{1}{2})x}{2\pi\sin(x/2)} + \frac{1}{Nx} + \ldots \sim \frac{1}{N}\delta'(Nx)$$

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- Example: Exponential(Gelb-Tadmor): $\sigma(\theta_k) = \theta_k \cdot e^{\frac{c}{\theta_k(\theta_k-1)}}$

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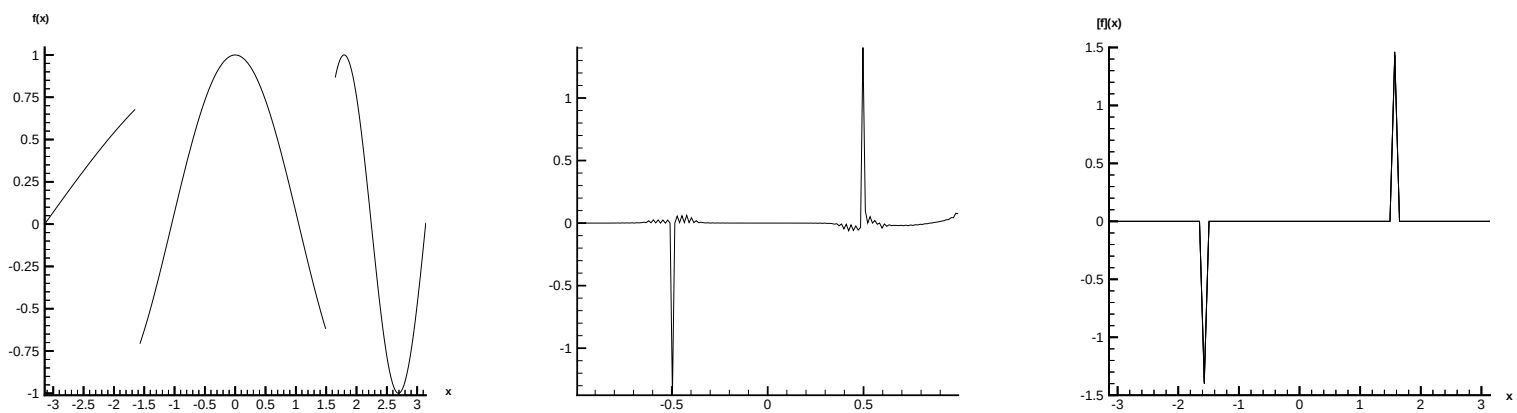

left: reconstructed f; center: Minmod detection $mm\{S_N^{\sigma=lin}, S_N^{exp}\}$ right: enhanced detection

Noisy data: $\widehat{f}(k) \sim [f](c)\frac{e^{-ikc}}{2\pi ik} + \widehat{\mathbf{n}}(k)$

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(i) if $\sqrt{\eta\lambda}N \ll 1$: the pointwise error $\epsilon_N := \frac{\log(N)}{N}$

(ii) if $\frac{1}{N} \lesssim \sqrt{\eta\lambda} \ll 1$: the pointwise error $\epsilon_\eta := \sqrt{\eta\lambda}\log\left(\sqrt{\eta\lambda}\right)$.

Noisy data: $f(x) = \overbrace{\mathbf{r}(x)}^{regular} + \overbrace{\mathbf{j}(x)}^{jump} + \overbrace{\mathbf{n}(x)}^{noise}$

- $(BV, L^2)-$ minimization:

$$\min_{\sum_k \frac{\sigma(\theta_k)}{N\theta_k}=1} k_0 \|S_N^\sigma(\mathbf{r})(x)\|_{BV} + \|S_N^\sigma(\mathbf{j})(x)\|_{L^2}^2 + \lambda \|S_N^\sigma(\mathbf{n})(x)\|_{L^2}^2$$

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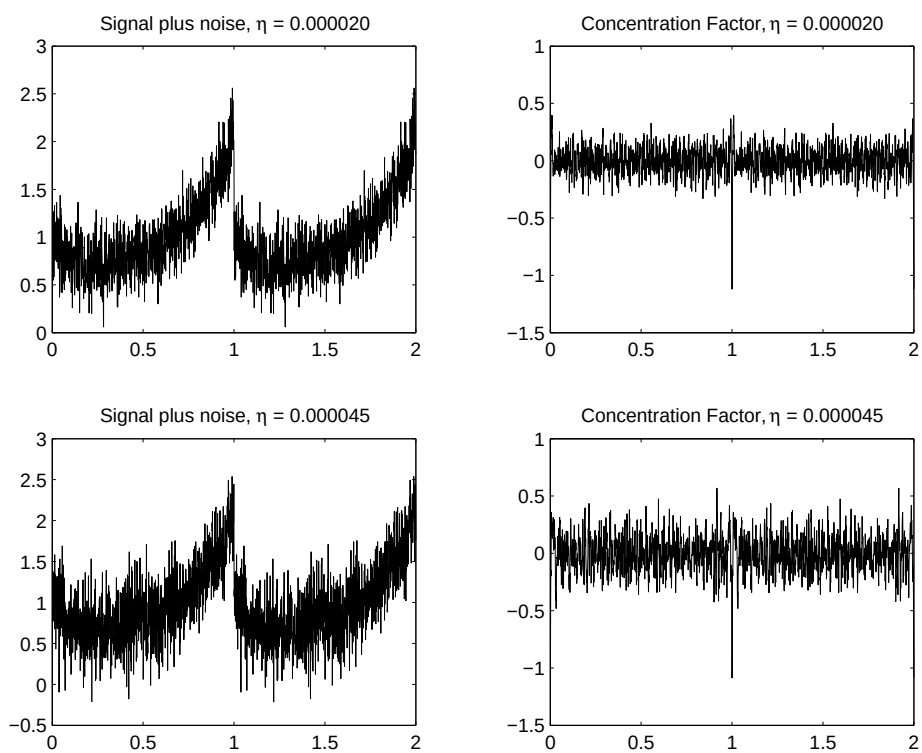

Incomplete data: $\widehat{f}(k),\ k \in \Omega \subset \{-N, \dots N\}$

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$$J_f(x) = \overbrace{\sum_{k \in \Omega} \widetilde{f}(k) e^{ikx}}^{\text{prescribed data}} + \overbrace{\sum_{k \notin \Omega} \widetilde{J}(k) e^{ikx}}^{\text{missing data}}.$$

- $\{\widetilde{J}(k) | k \notin \Omega\}$ — sought as minimizers of the $\|J(x)\|_{BV}$:

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- <u>The rationale:</u> Missing $\left\{\widetilde{J_f}(k)\right\}_{|k \notin \Omega}$ complement the prescribed $\left\{\widetilde{f}(k)\right\}_{|k \in \Omega}$ as the approximate Fourier coefficients of $(\widehat{J_f})_k$

Left: the original image; center: recovered by the back projection method; right: recovered edges by the compressed sensing edge detection method.

Here, 3000 Fourier coefficients are uniformly and randomly distributed over the 128^2 gridpoints. We construct the spectral data of the edges captured by the Sobel edge detection method.

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REF: The Computation of Gibbs Phenomenon [Acta Numerica 2007]

Multiscale Decomposition of Images in (BV,L^p)*

* with Prashant Athavale (Maryland) Suzanne Nezzar (Stockton)
and Luminita Vese (UCLA)

Image processing: (X, Y)-multiscale decompositions

$f \in L^2(\Omega)$, (grayscale) $\mathbf{f} = (f_1, f_2, f_3) \in L^2(\Omega)^3$, (colored): $\Omega \subset I\!R^2$

Noticeable features: edges quantified in $X = BV, ..., \|u\|_X := \int_\Omega \phi(D^p u), ..$

Other features: blur, patterns, noise, texture,...: in $Y = L^2, ..., L^1, ..$

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Images in intermediate spaces 'between' $X = BV$ and $Y = L^2, \ldots$

- $T : Y \hookrightarrow Y$ is a blurring operator:

$$\mathcal{Q}_T(f, \lambda; X, Y) := \inf_{u \in X(\Omega)} \left\{ \|u\|_X + \lambda \|f - Tu\|_Y^2 \right\}$$

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Example: λ as a *fixed* threshold of noisy part: $\|f - Tu\|_{L^2} \leq \sigma_\lambda$

$$f = Tu_\lambda + v_\lambda, \quad [u_\lambda, v_\lambda] = \arg\inf_{Tu+v=f} \mathcal{Q}(f, \lambda; X, Y).$$

- u_λ extracts the *edges*; v_λ captures *textures*

Multiscale decompositions — blurry and noisy images

- Distinction is scale dependent – 'texture' at a λ-scale consists of significant edges when viewed under a refined 2λ-scale

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This process can continue...

$$[u_{j+1}, v_{j+1}] = \arg\inf_{Tu_{j+1}+v_{j+1}=v_j} \mathcal{Q}_T(v_j, \lambda_{j+1}), \quad \lambda_j := \lambda_0 2^j, \ j = 0, 1, \ldots$$

After k hierarchical step:

$$\begin{aligned} f &= Tu_0 + v_0 = \\ &= Tu_0 + Tu_1 + v_1 = \\ &= \ldots\ldots \qquad\qquad = \\ &= Tu_0 + Tu_1 + \cdots + Tu_k + v_k. \end{aligned}$$

Hierarchical expansion and energy decomposition

- Iterated Tikhonov regularization – expansion of denoised/deblurred image:

$$u \cong \sum_{j \geq 0}^{k} u_j \quad \Longleftarrow \quad f = Tu_0 + Tu_1 + ... + Tu_{k-1} + Tu_k + v_k$$

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Example. $f = \chi_{B_R} : v_j = \frac{1}{\lambda_j R}\chi_{B_R} - \frac{1}{\lambda_j R}\frac{|B_R|}{|\Omega \setminus B_R|}\chi_{\Omega \setminus B_R} \sim 2^{-j}$ universal

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Three basic questions about multiscale decompositions

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Three basic questions about multiscale decompositions

- Where to start: $u \cong \sum_{j\geq j_0} u_j : \quad \frac{1}{2} < 2^{j_0}\lambda_0\|T^* f\|_* \leq 1$

Hierarchical expansion and energy decomposition

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- Where to start: $u \cong \sum_{j\geq j_0} u_j : \quad \frac{1}{2} < 2^{j_0}\lambda_0\|T^* f\|_* \leq 1$

- When to stop? $u \cong \sum_{j\geq j_0}^{k} u_j$

Hierarchical expansion and energy decomposition

- Iterated Tikhonov regularization – expansion of denoised/deblurred image:

$$u \cong \sum_{j\geq 0}^{k} u_j \quad \Longleftarrow \quad f = Tu_0 + Tu_1 + \ldots + Tu_{k-1} + Tu_k + v_k$$

Example. $f = \chi_{B_R} : v_j = \frac{1}{\lambda_j R}\chi_{B_R} - \frac{1}{\lambda_j R}\frac{|B_R|}{|\Omega\setminus B_R|}\chi_{\Omega\setminus B_R} \sim 2^{-j}$ universal

- Energy decomposition: $\sum_{j=0}^{\infty}\left[\frac{1}{\lambda_j}\|u_j\|_X + \|Tu_j\|_{L^2}^2\right] = \|f\|_{L^2}^2,\ f \in X$

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- Why dyadic scales — $\lambda_0 < \lambda_1 < \ldots < \lambda_k, \quad \lambda_k \sim 2^k$?

Examples of multiscale image decompositions

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- Multiscale $X = BV,\ Y = L^2$-decomposition

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- Mumford-Shah: $X = SBV,\ Y = L^2$-decomposition

- Osher-Vese: $X = BV,\ Y = BV^*$ ...

- Multiplicative noise: $\inf\limits_{u \in BV_+(\Omega)} \left\{ \|u\|_{BV(\Omega)} + \lambda \left\| \frac{f}{u} - 1 \right\|_{L^2(\Omega)}^2 \right\}$

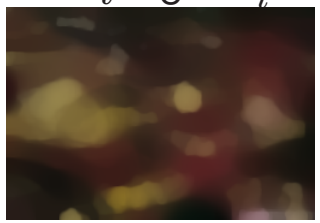
(BV, L^2)-decomposition of Barbara

$$u_{j+1} - \frac{1}{2\lambda_{j+1}} \mathrm{div}\left(\frac{\nabla u_{j+1}}{|\nabla u_{j+1}|}\right) = -\frac{1}{2\lambda_j} \mathrm{div}\left(\frac{\nabla u_j}{|\nabla u_j|}\right)$$

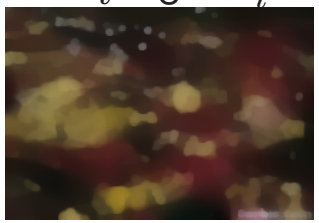
(BV, L^2)-decomposition of colored images

f

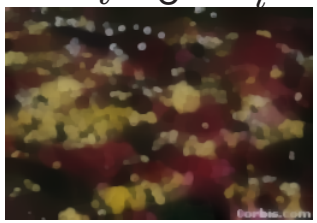
$\sum_{i=0}^{2} u_{\lambda_i}$



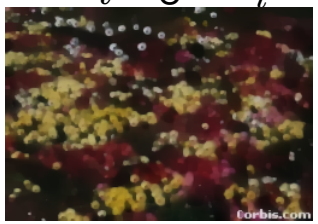
$\sum_{i=0}^{3} u_{\lambda_i}$



$\sum_{i=0}^{4} u_{\lambda_i}$



$\sum_{i=0}^{5} u_{\lambda_i}$



$\sum_{i=0}^{6} u_{\lambda_i}$

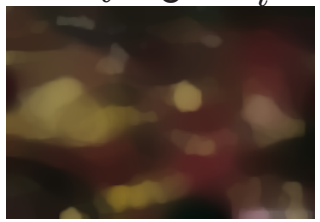
$\sum_{i=0}^{7} u_{\lambda_i}$

$\sum_{i=0}^{8} u_{\lambda_i}$

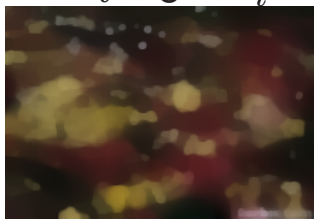
(BV, L^2)-decomposition of colored images

f

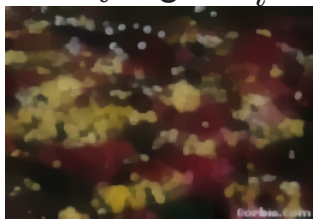
$\sum_{i=0}^{2} u_{\lambda_i}$



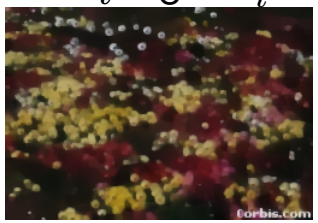
$\sum_{i=0}^{3} u_{\lambda_i}$



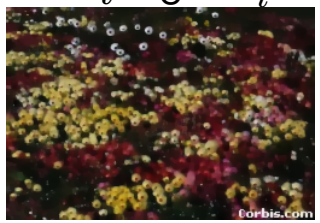
$\sum_{i=0}^{4} u_{\lambda_i}$



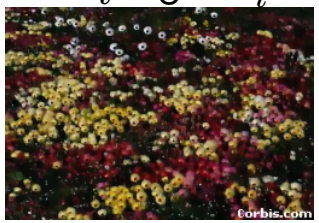
$\sum_{i=0}^{5} u_{\lambda_i}$



$\sum_{i=0}^{6} u_{\lambda_i}$

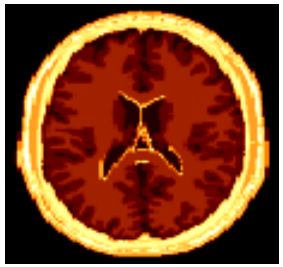


$\sum_{i=0}^{7} u_{\lambda_i}$

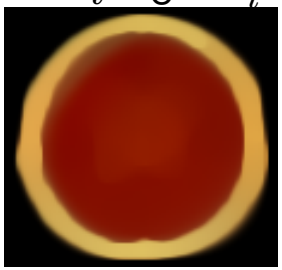


$\sum_{i=0}^{8} u_{\lambda_i}$

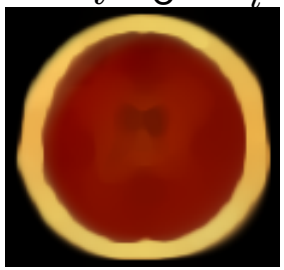
f



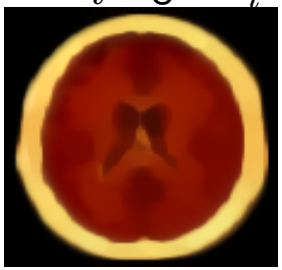
$\sum_{i=0}^{2} u_{\lambda_i}$



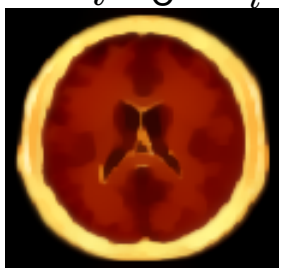
$\sum_{i=0}^{3} u_{\lambda_i}$



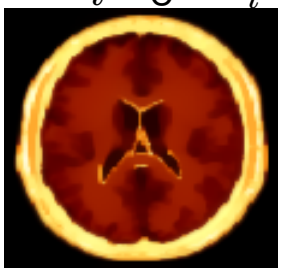
$\sum_{i=0}^{4} u_{\lambda_i}$



$\sum_{i=0}^{5} u_{\lambda_i}$



$\sum_{i=0}^{6} u_{\lambda_i}$



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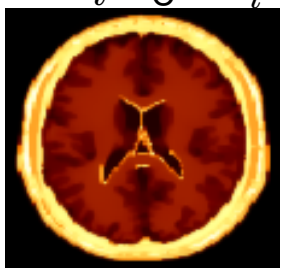
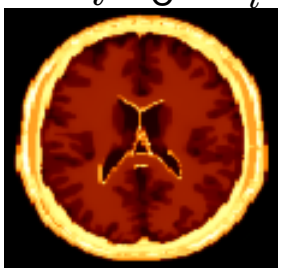
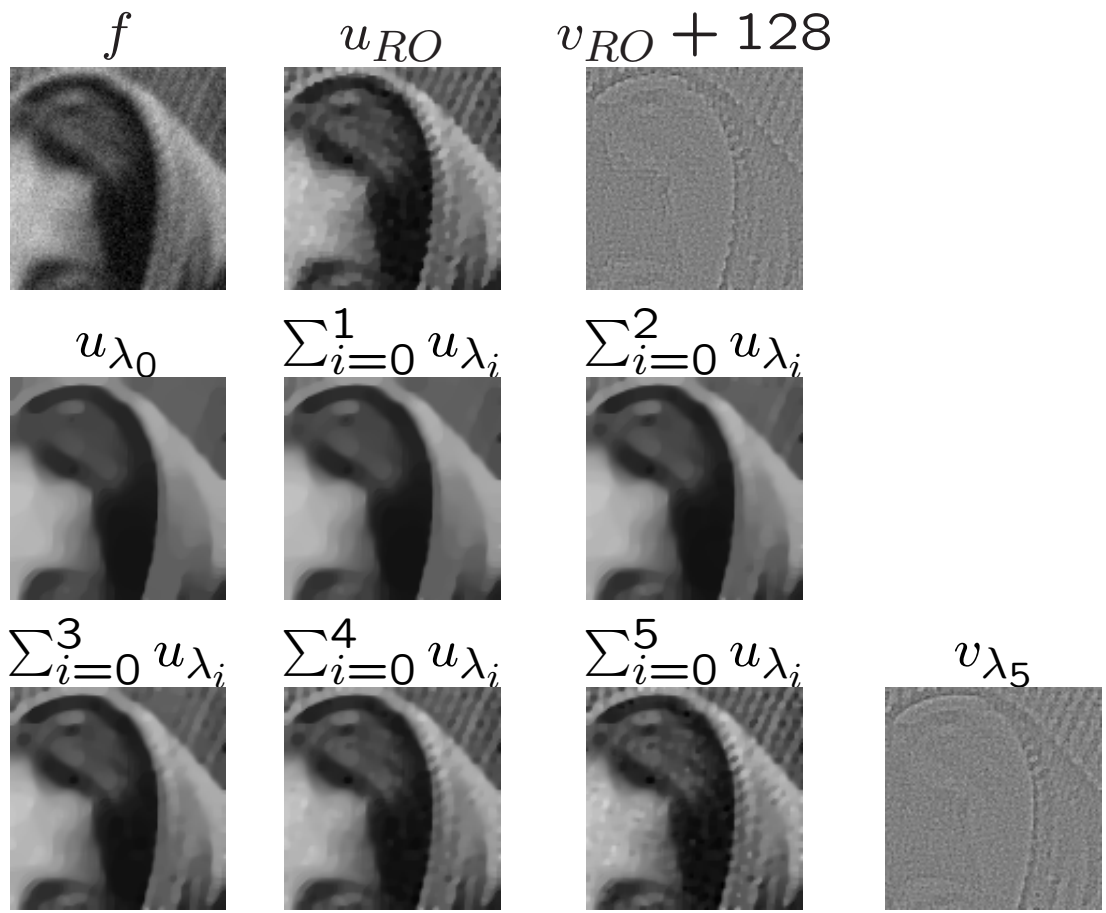


$\sum_{i=0}^{8} u_{\lambda_i}$



Multiscale $\mathcal{Q}_T(BV, L^2)$ deblurring+denoising



1st row, from left to right: blurry-noisy version f, and Rudin-Osher restoration u_{RO}, $v_{RO} = f - u_{RO} + 128$, rmse =0.2028. RO parameters $\lambda = 0.5$, $h = 1$, $\triangle t = 0.025$. The other rows, left to right: the hierarchical recovery of u from the same blurry-noisy initial image f using 6 steps. Parameters: $\lambda_k = .01 * 2^{k+1}$, $\Delta t = 0.025$, $h = 1$; rmse= 0.2011: avoids staircasing

(BV, L_1) vs. (BV, L_1^2)-decomposition

$$\sum^{k} u_j = f + \frac{|u_k - f|}{2\lambda_k \|u_k - f\|_{L_1}} \nabla\Big(\frac{\nabla u_k}{|\nabla u_k|}\Big), \;\; \lambda_k = 2\lambda_{k-1}.$$

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Original noisy image

(BV,L_1^2)

λ=6.5536

(BV,L_1^2)

λ=13.1072

(BV,L_1^2)

λ=26.2144

(BV,L_1^2)

λ=52.4288

(BV, L_1) vs. (BV, L_1^2)-decomposition enhanced

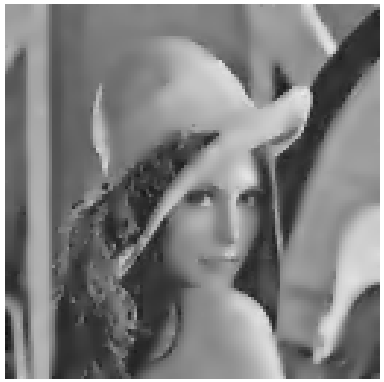
$$\sum^{k} u_j = f + \frac{|u_k - f|\, g(|G * \nabla u_k|)}{2\lambda_k \|u_k - f\|_{L_1}} \nabla\Big(\frac{\nabla u_k}{|\nabla u_k|}\Big), \;\; \lambda_k = 2\lambda_{k-1}.$$

(BV, L_1) vs. (BV, L_1^2)-decomposition enhanced

$$\sum^{k} u_j = f + \frac{|u_k - f|\, g(|G * \nabla u_k|)}{2\lambda_k \|u_k - f\|_{L_1}} \nabla\left(\frac{\nabla u_k}{|\nabla u_k|}\right), \quad \lambda_k = 2\lambda_{k-1}.$$

Original noisy image

(BV,L_1^2) with g(|G*Du|)



λ=6.5536

(BV,L_1^2) with g(|G*Du|)

λ=13.1072

(BV,L_1^2) with g(|G*Du|)

λ=26.2144

(BV,L_1^2) with g(|G*Du|)

λ=52.4288

(BV, L_1) vs. (BV, L_1^2) -decomposition enhanced

(BV, L_1) vs. (BV, L_1^2) -decomposition enhanced

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Original noisy image

(BV,L_1) with g(|G*Du|)

λ=6.5536

(BV,L_1) with g(|G*Du|)

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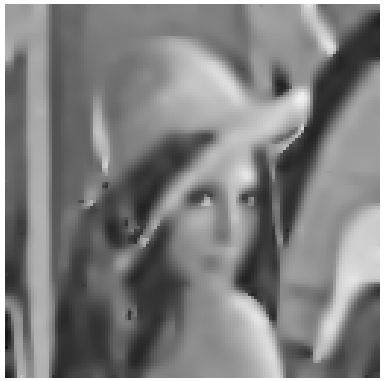
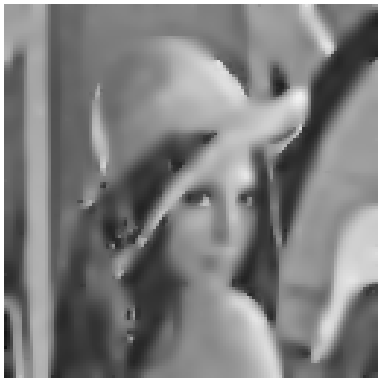


λ=26.2144

(BV,L_1) with g(|G*Du|)



λ=52.4288

(BV, L_1) vs. (BV, L_1^2) in normal direction

(BV, L_1) vs. (BV, L_1^2) in normal direction

$$\sum^{k} u_j = f + \frac{|u_k - f|\, g(|G * \nabla u_k|)|\nabla u_k|}{2\lambda_k \|u_k - f\|_{L_1}} \nabla\left(\frac{\nabla u_k}{|\nabla u_k|}\right), \quad \lambda_k = 2\lambda_{k-1}.$$

Original noisy image

(BV,L_1^2) with g(|G*Du|)|Du|

λ=6.5536

(BV,L_1^2) with g(|G*Du|)|Du|

λ=13.1072

(BV,L_1^2) with g(|G*Du|)|Du|

λ=26.2144

(BV,L_1^2) with g(|G*Du|)|Du|

λ=52.4288

Multiscale (SBV, L^2) decomposition

- The regularized Mumford-Shah by Ambrosio and Tortorelli (AT):

$$\mathcal{AT}_\rho(f,\lambda) \;:=\; \inf_{\{u,v,w \;|\; u+v=f\}} \left\{ \mu\sqrt{\int_\Omega (w^2+\rho\epsilon_\rho)|\nabla u|^2 dx} + \right.$$

$$\rho\epsilon_\rho \to 0, \rho \downarrow 0 \qquad \left. +\rho\|\nabla w\|^2_{L^2(\Omega)} + \frac{\|w-1\|^2_{L^2(\Omega)}}{4\rho} + \lambda\|v\|^2_{L^2(\Omega)} \right\},$$

$$[u_{j+1}, v_{j+1}] = \arg\inf_{u,v,w,u+v=v_j} \mathcal{AT}_\rho(v_j,\lambda_j), \quad \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

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- Energy scaling: $\displaystyle\sum_{j=0}^{\infty}\left[\frac{1}{\lambda_j}|u_j|_{H^1_{w_j}(\Omega)} + \|u_j\|^2_{L^2}\right] \le \|f\|^2_{L^2}$

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Hierarchical decomposition: $f \cong \displaystyle\sum_{j=0} u_j, \;\; \|f - \sum_{j=0}^{k} u_j\|_{H^{-1}_{w_k}(\Omega)} = \frac{\mu}{\lambda_{k+1}}$

Multiscale (SBV, L^2) decomposition

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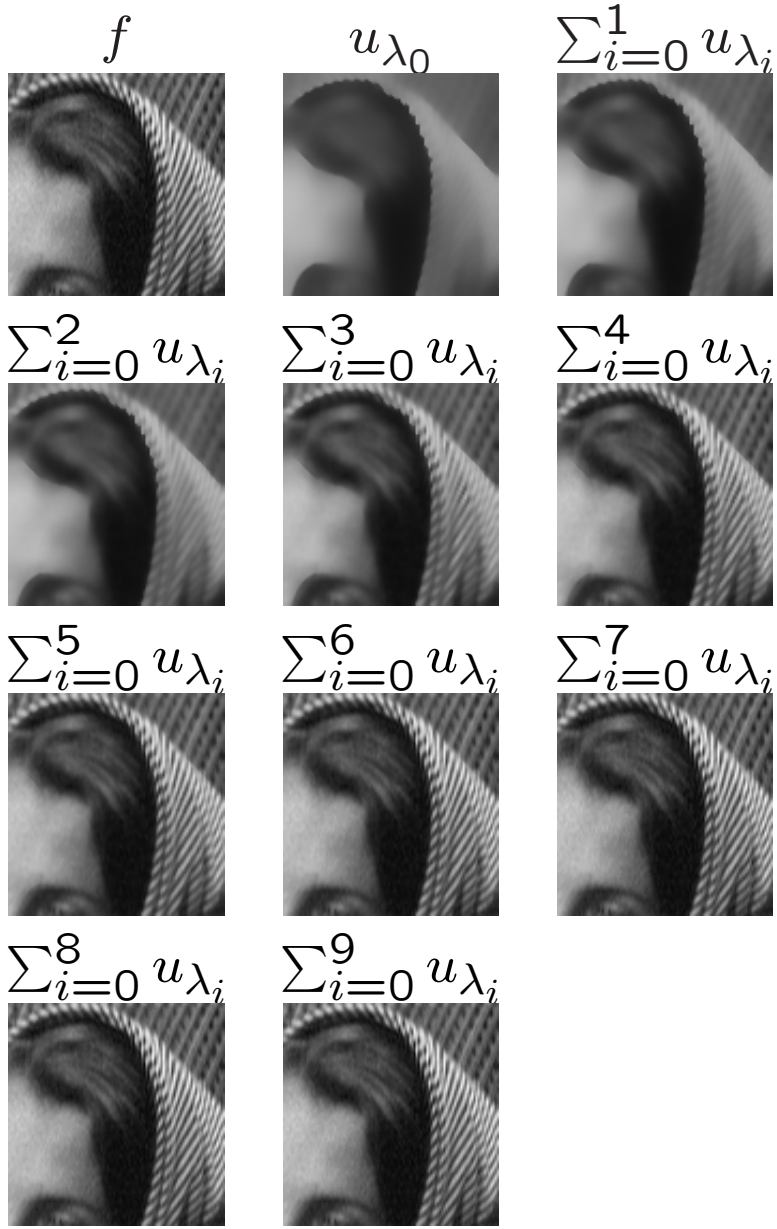
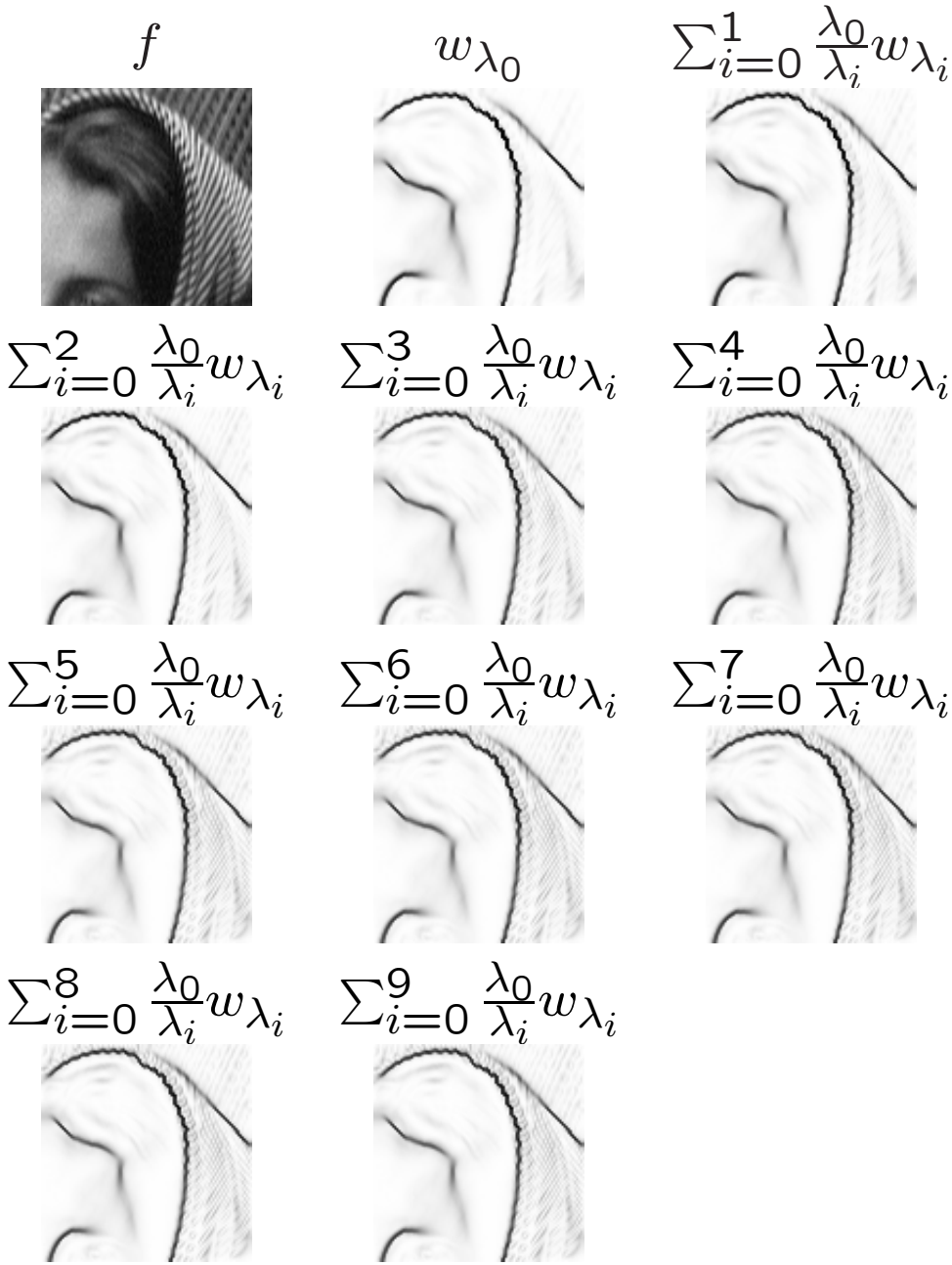
Hierarchical decomposition: $\displaystyle f \cong \sum_{j=0} u_j, \;\; \|f - \sum_{j=0}^{k} u_j\|_{H^{-1}_{w_k}(\Omega)} = \frac{\mu}{\lambda_{k+1}}$

- Discrete AT-weights, $1-w_j$: edge detectors

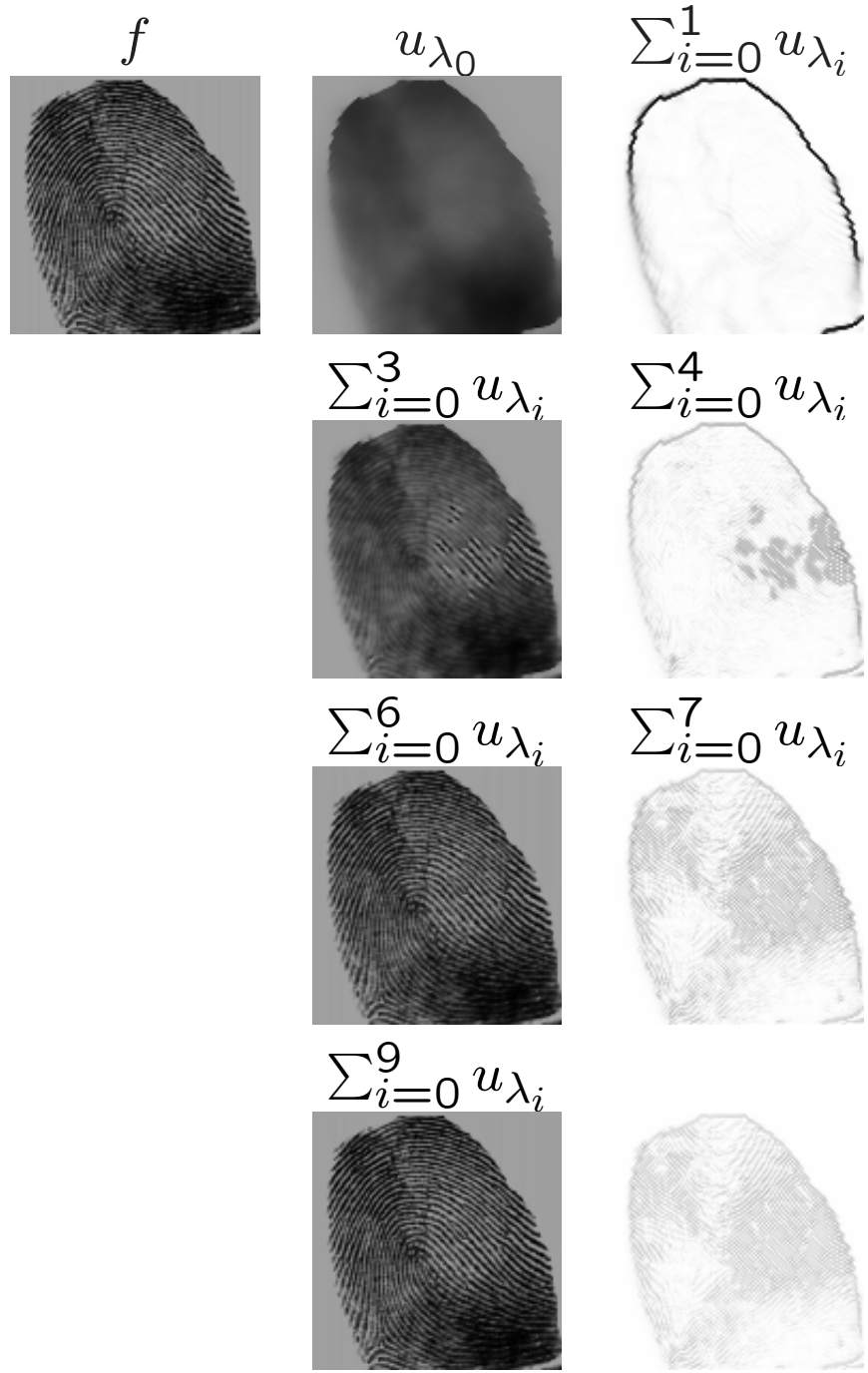
$$1 - w_j(x,y) \approx \begin{cases} 0, & \text{in regions of smoothness} \\ 1, & \text{near edges} \end{cases}$$



Ambrosio-Tortorelli hierarchical decomposition in 10 steps. Parameters: $\lambda_0 = .25$, $\mu = 5$, $\rho = .0002$, and $\lambda_k = 2^k \lambda_0$.

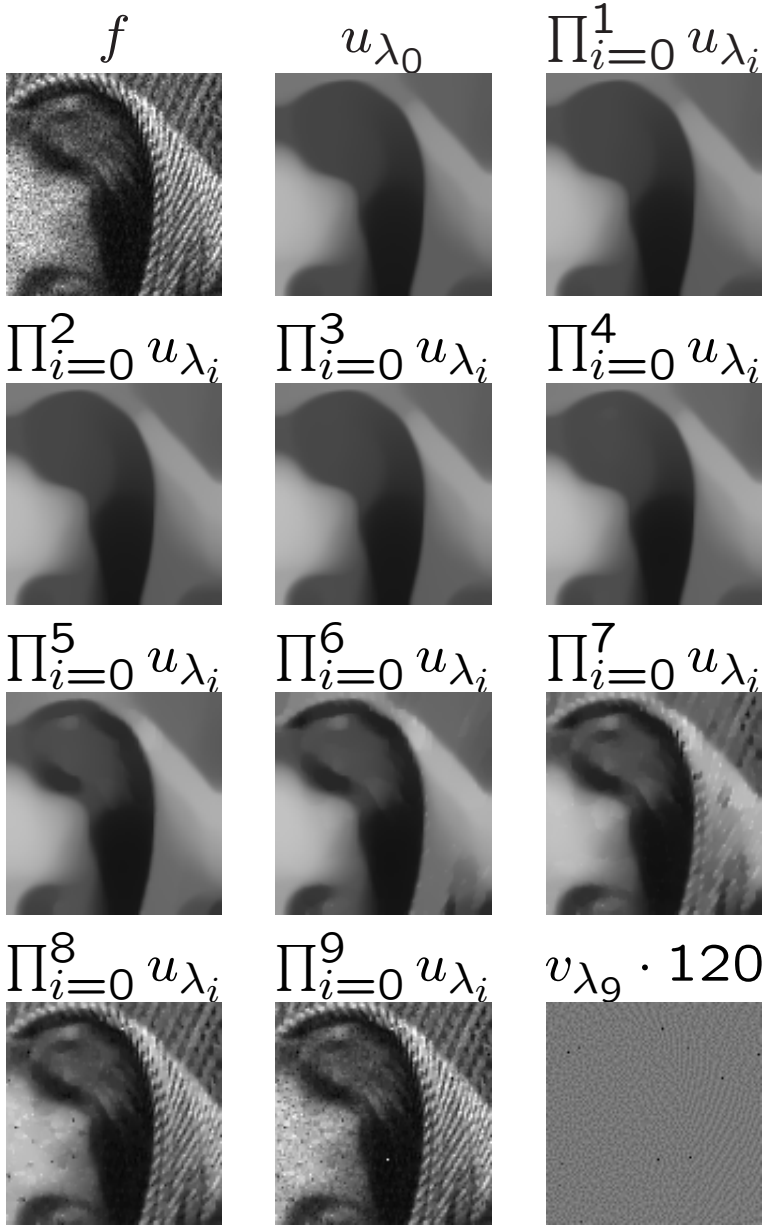


The weighted sum of the w_i's using AT decomposition in 10 steps.



The AT hierarchical decomposition of a fingerprint.

Remove multiplicative noise



The recovery of u given an initial image of a woman with multiplicative noise, for 10 steps. Parameters: $\lambda_0 = .02$, and $\lambda_k = 2^k \lambda_0$.

Evolution in inverse scale space

$$u_{j+1} - \frac{1}{2\lambda_{j+1}} \mathrm{div}\left(\frac{\nabla u_{j+1}}{|\nabla u_{j+1}|}\right) = -\frac{1}{2\lambda_j} \mathrm{div}\left(\frac{\nabla u_j}{|\nabla u_j|}\right)$$

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- Recursive; in fact, telescoping....

$$\sum_{j=1}^{k} u_j - f = \frac{1}{2\lambda_k}\mathrm{div}_x\left(\frac{\nabla_x u_k}{|\nabla_x u_k|}\right) \mapsto \int_{s=0}^{t} u(s)ds - f = \frac{1}{\lambda(t)}\mathrm{div}_x\left(\frac{\nabla_x u(t)}{|\nabla_x u(t)|}\right)$$

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... compared w/Perona-Malik: $u_t = \frac{1}{\lambda(t)}\text{div}_x\left(\frac{\nabla_x u(t)}{|\nabla_x u(t)|}\right)$

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- The role of $\lambda(t)$? dictates the size of texture: $\|U(\cdot,t) - f\|_{BV^*}$!

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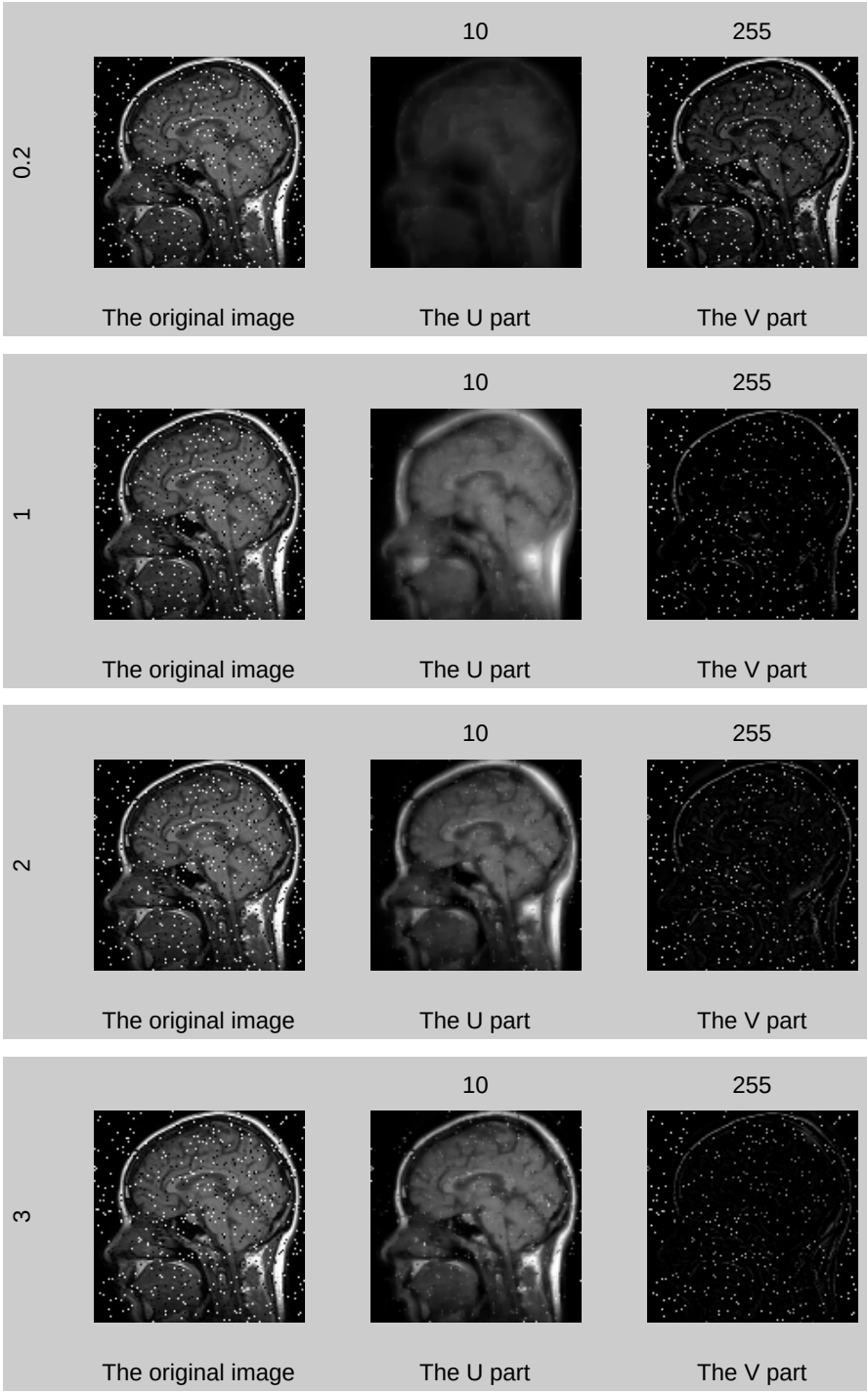
- The role of $\lambda(t)$? dictates the size of texture: $\|U(\cdot,t) - f\|_{BV^*}$!

- Restrict the diffusion in directions orthogonal to edges

$$\int_{s=0}^{t} u(x,s)ds - f(x) = \frac{1}{\lambda(t)}g(G * \nabla_x u(x,t))|\nabla_x u(x,t)| \cdot \text{div}_x\left(\frac{\nabla_x u(x,t)}{|\nabla_x u(x,t)|}\right)$$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0.2
The original image
The U part
The V part
10
255
1
The original image
The U part
The V part
10
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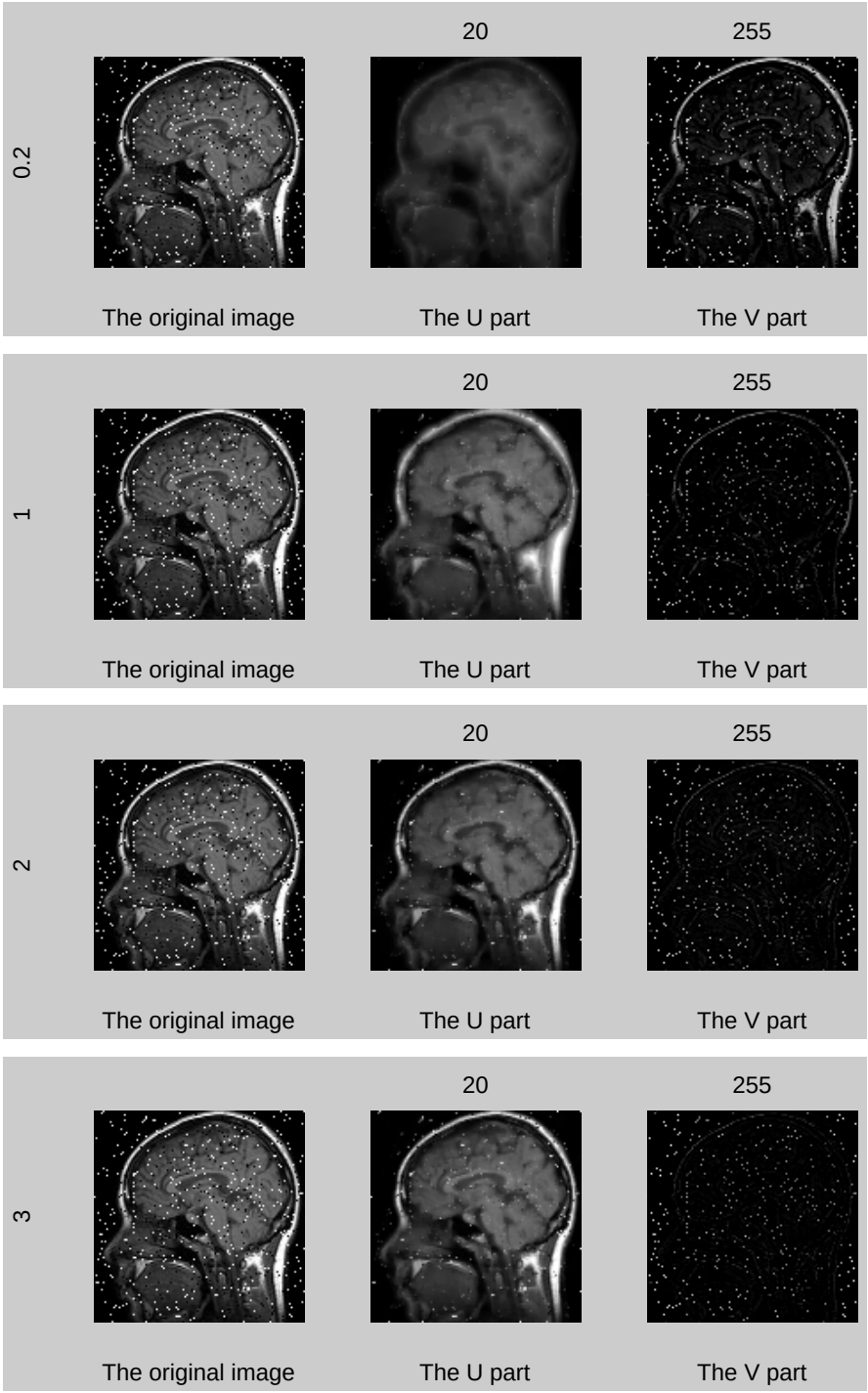
2
The original image
The U part
The V part
10
255
3
The original image
The U part
The V part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255
0.2
The original image
The U part
The V part
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1
The original image
The U part
The V part
20
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2
The original image
The U part
The V part
20
255
3
The original image
The U part
The V part

THANK YOU