Numerical Methods for Modern Traffic Flow Models

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Traffic Flow Models with Arrhenius Look-Ahead Dynamics

[M.J. Lighthill, G.B. Whitham; 1955] proposed the simplest deterministic continuum traffic flow model:

$$u_t + f(u)_x = 0, \qquad f(u) = uV(u), \qquad V(u) = V_{\mathsf{m}}(1-u)$$

$u(x,t)$: a relative car density ($0 \le u \le 1$)

$V_{\mathsf{m}} \equiv$ const: the maximum possible speed

$V(u)$: dimensionless velocity function that depends only on u

[A. Sopasakis, M.A. Katsoulakis; 2006] extended the Lighthill-Whitham model to a more realistic one by taking into account interactions with other vehicles ahead ("look-ahead" rule).

The resulting scalar conservation law with a global flux is:

$$u_t + F(u)_x = 0, \qquad F(u) = V_{\mathsf{m}} u(1-u)\exp(-J \circ u),$$

where the contribution of short range interactions to the flux is:

$$J \circ w(x) := \int_x^{x+\gamma} J(y-x)w(y)\,dy$$

$\gamma \equiv \text{const} > 0$: look-ahead distance

J: interaction potential

The simplest interaction potential is the constant one:

$$J(r) = \begin{cases} 1/\gamma, & 0 < r < \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$J \circ u(x) := \frac{1}{\gamma} \int_{x}^{x+\gamma} u(y)\, dy$$

Introducing the anti-derivative, $U := \int_{-\infty}^{x} u(\xi, t)\, d\xi$, we rewrite the global flux equation as:

$$u_t + F(u, U)_x = 0$$

$$F(u, U) = V_{\mathrm{m}} u(1-u) \underline{\underline{\exp\left(-\frac{U(x+\gamma) - U(x)}{\gamma}\right)}}$$

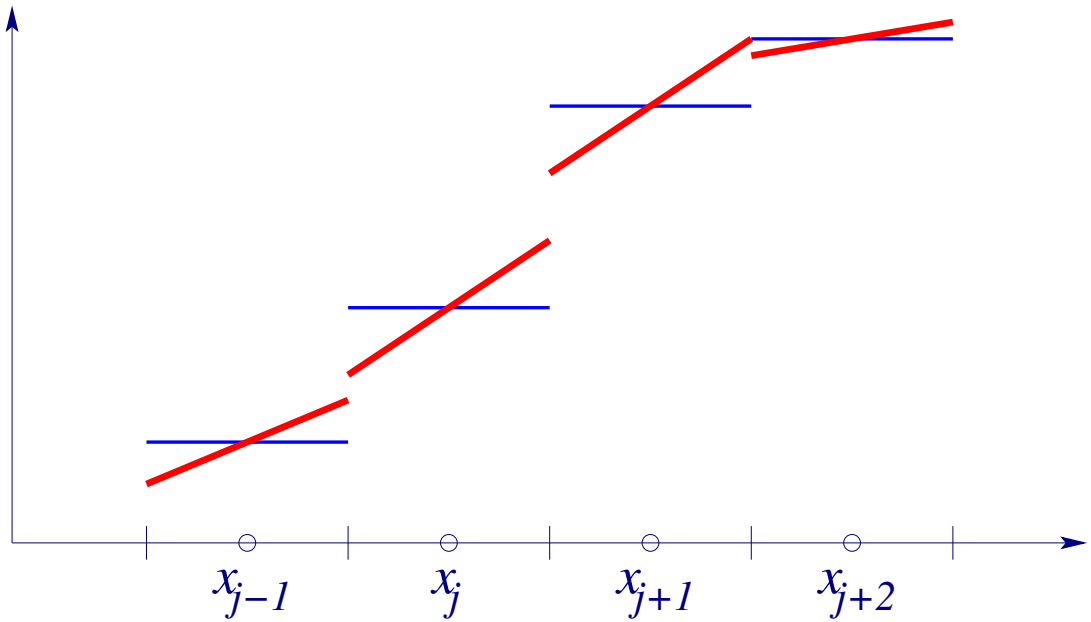
Finite-Volume Schemes

1-D System: $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$

$\bar{\mathbf{u}}_j^n \approx \dfrac{1}{\Delta x} \displaystyle\int_{C_j} \mathbf{u}(x, t^n)\, dx$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

This solution is approximated by a piecewise linear (conservative, second-order accurate, non-oscillatory) reconstruction:

$\tilde{\mathbf{u}}^n(x) = \bar{\mathbf{u}}_j^n + \mathbf{s}_j^n(x - x_j)$ for $x \in C_j$



For example,

$$\boxed{\mathbf{s}_j^n = \text{minmod}\left(\theta\frac{\bar{\mathbf{u}}_j^n - \bar{\mathbf{u}}_{j-1}^n}{\Delta x}, \frac{\bar{\mathbf{u}}_{j+1}^n - \bar{\mathbf{u}}_{j-1}^n}{2\Delta x}, \theta\frac{\bar{\mathbf{u}}_{j+1}^n - \bar{\mathbf{u}}_j^n}{\Delta x}\right)} \quad \theta \in [1, 2],$$

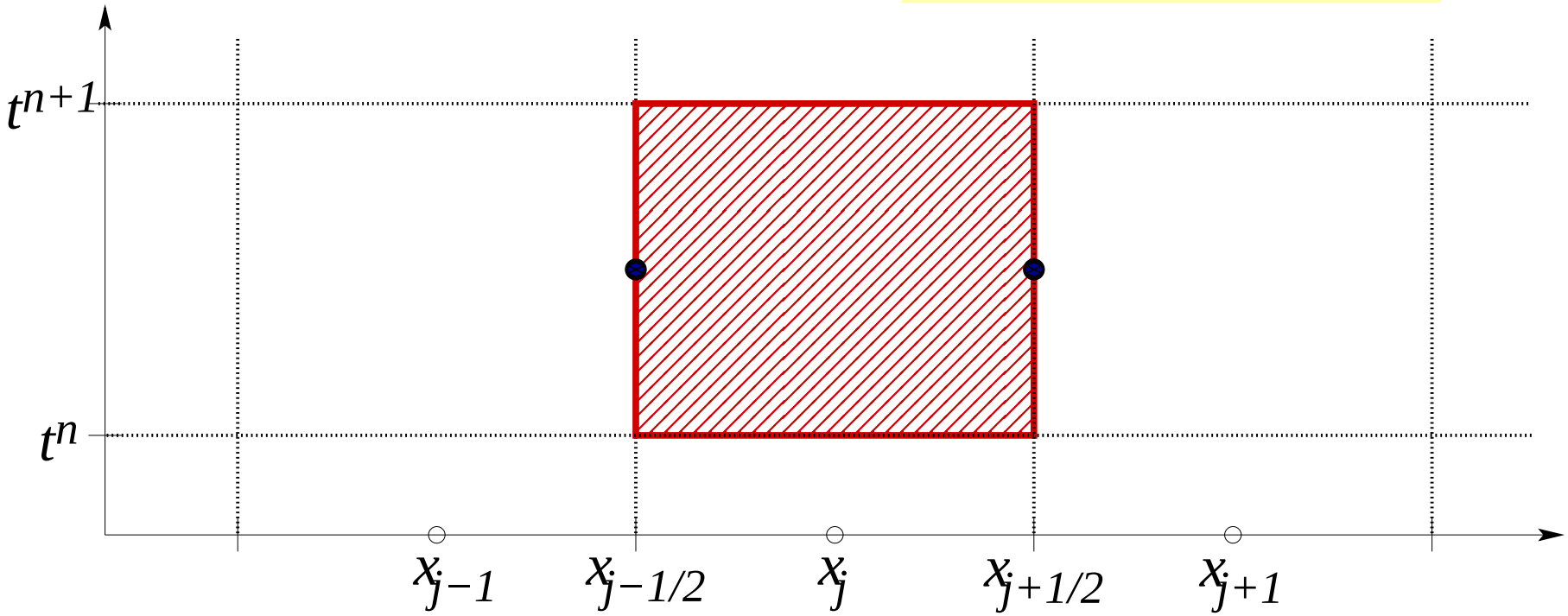
where the minmod function is defined as:

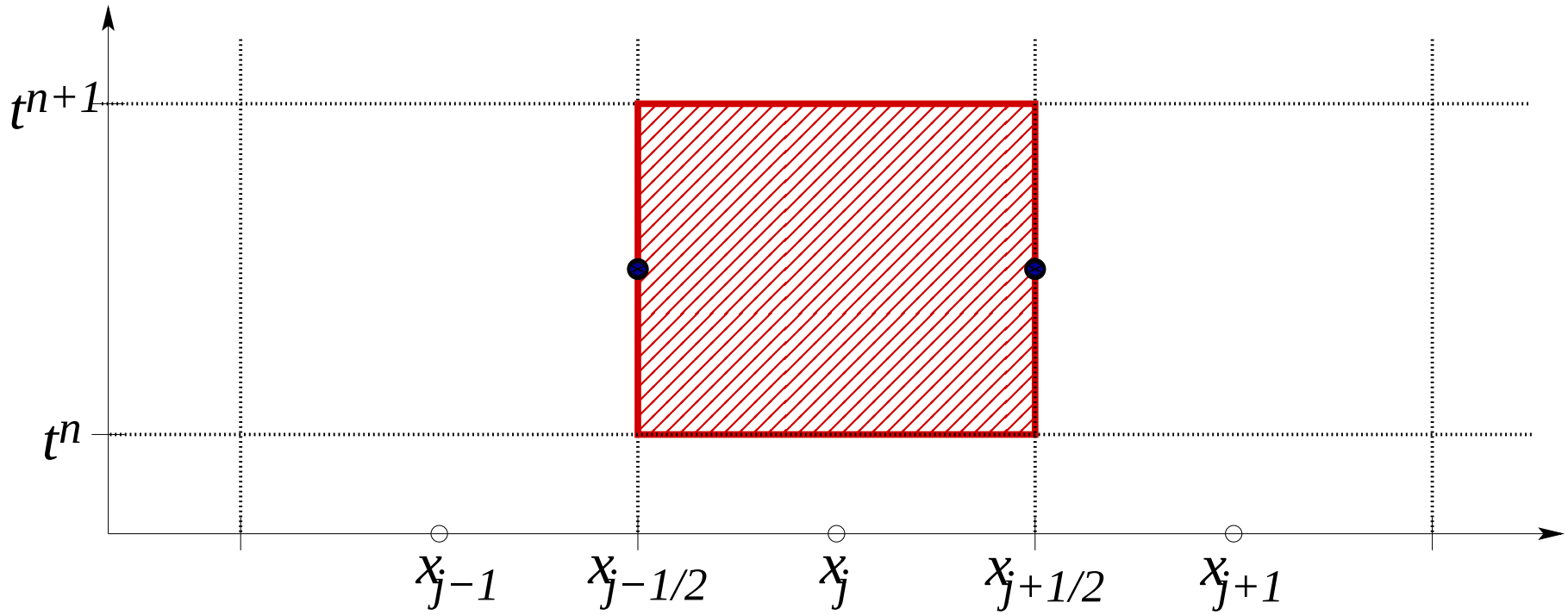
$$\boxed{\text{minmod}(z_1, z_2, \ldots) := \begin{cases} \min_j\{z_j\}, & \text{if} \quad z_j > 0 \quad \forall j, \\ \max_j\{z_j\}, & \text{if} \quad z_j < 0 \quad \forall j, \\ 0, & \text{otherwise.} \end{cases}}$$

Godunov-type upwind schemes are designed by integrating

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$$

over the space-time control volumes $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$





$$\bar{\mathbf{u}}_j^{n+1} = \bar{\mathbf{u}}_j^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t)) - \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t)) \right] dt$$

In order to evaluate the flux integrals on the RHS, one has to (approximately) solve the generalized Riemann problem.

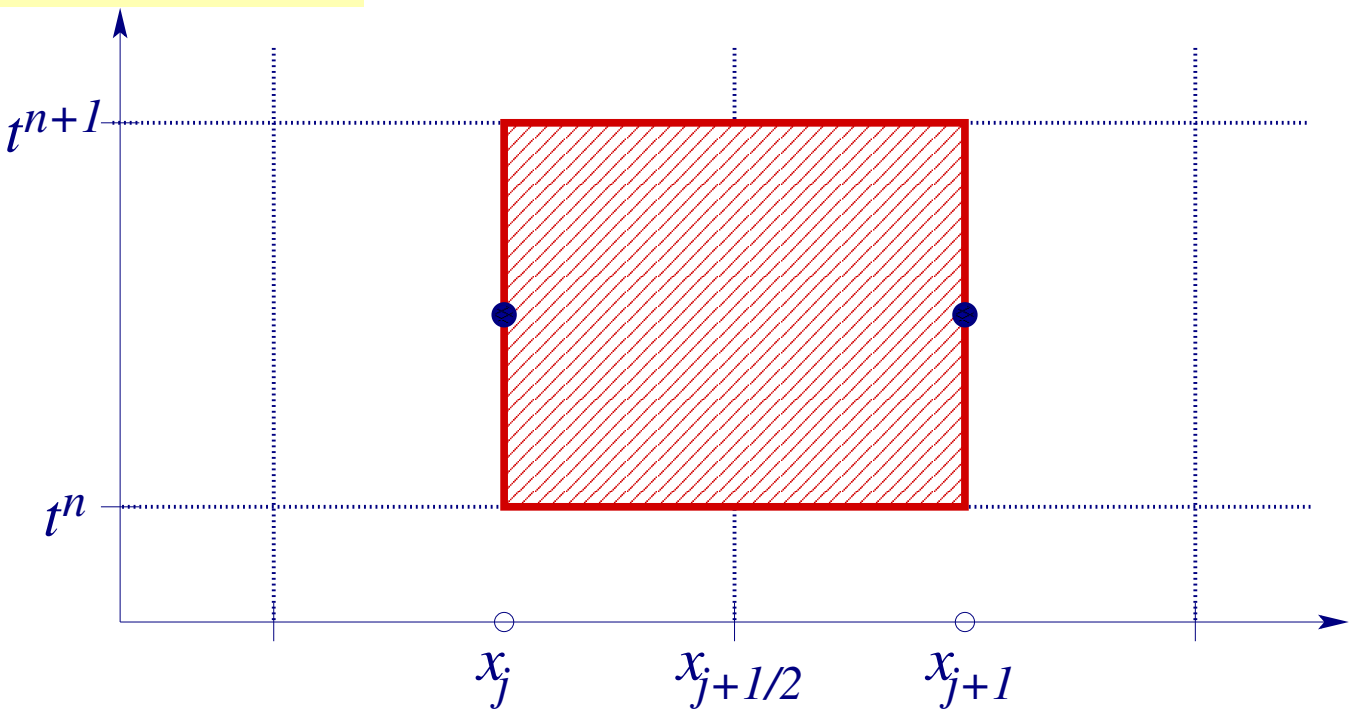
This may be hard or even impossible...

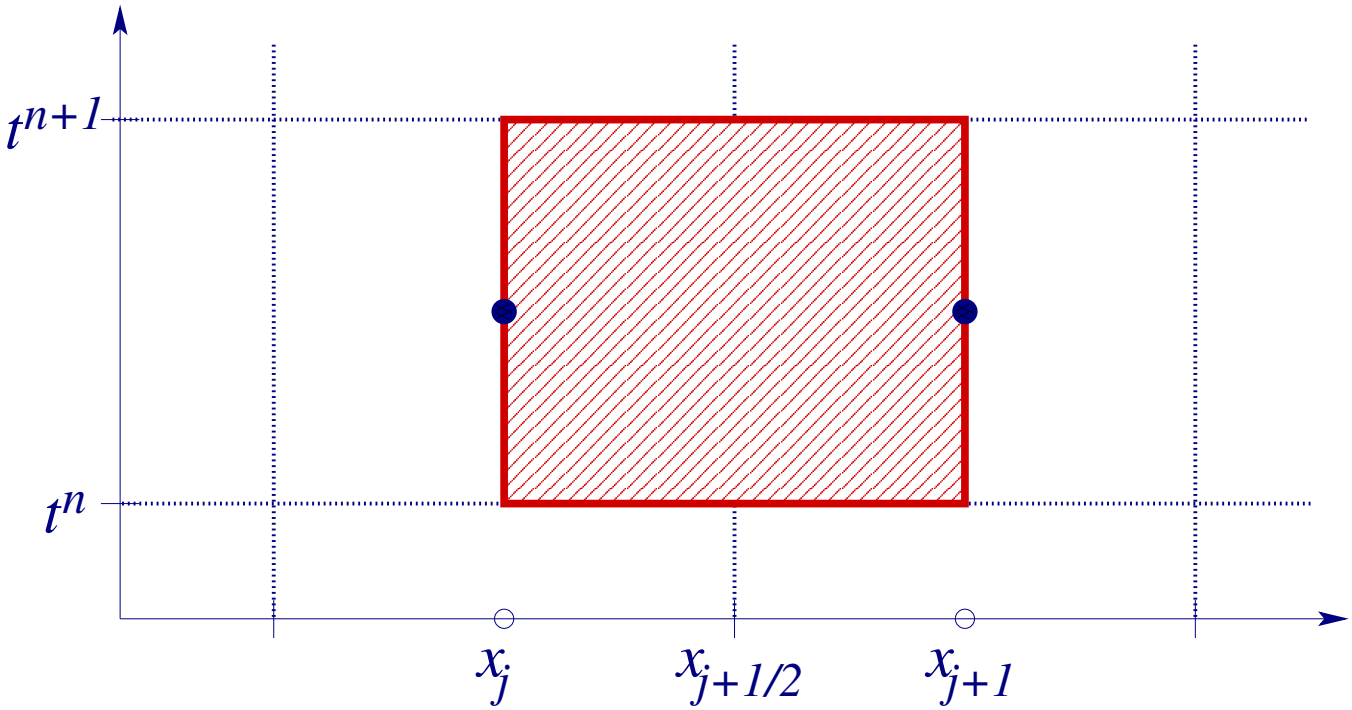
Nessyahu-Tadmor Scheme

The Nessyahu-Tadmor [H. Nessyahu, E. Tadmor; 1990] scheme is a central Godunov-type scheme. It is designed by integrating

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}$$

over the different set of staggered space-time control volumes $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$ containing the Riemann fans





$$\bar{\mathbf{u}}_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \tilde{\mathbf{u}}^n(x)\, dx - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \Big[\mathbf{f}(\mathbf{u}(x_{j+1}, t)) - \mathbf{f}(\mathbf{u}(x_j, t))\Big] dt$$

Due to the finite speed of propagation, this can be reduced to:

$$\bar{\mathbf{u}}_{j+\frac{1}{2}}^{n+1} = \frac{\bar{\mathbf{u}}_j^n + \bar{\mathbf{u}}_{j+1}^n}{2} + \frac{\Delta x}{8}(\mathbf{s}_j^n - \mathbf{s}_{j+1}^n) - \frac{\Delta t}{\Delta x}\Big[\mathbf{f}(\mathbf{u}_{j+1}^{n+\frac{1}{2}}) - \mathbf{f}(\mathbf{u}_j^{n+\frac{1}{2}})\Big]$$

Values of $\mathbf{u}$ at $t = t^{n+\frac{1}{2}}$ are approximated using the Taylor expansion:

$$\mathbf{u}_j^{n+\frac{1}{2}} \approx \widetilde{\mathbf{u}}^n(x_j) + \frac{\Delta t}{2}\mathbf{u}_t(x_j, t^n)$$

- $\widetilde{\mathbf{u}}^n(x) = \bar{\mathbf{u}}_j^n + \mathbf{s}_j^n(x - x_j) \Longrightarrow \boxed{\widetilde{\mathbf{u}}^n(x_j) = \bar{\mathbf{u}}_j^n}$

- $\boxed{\mathbf{u}_t(x_j, t^n) = -\mathbf{f}_x(\bar{\mathbf{u}}_j^n)}$

The space derivatives $\mathbf{f}_x$ are computed using the (minmod) limiter:

$$\mathbf{f}_x(\bar{\mathbf{u}}_j^n) = \text{minmod}\left(\theta\,\frac{\mathbf{f}(\bar{\mathbf{u}}_j^n) - \mathbf{f}(\bar{\mathbf{u}}_{j-1}^n)}{\Delta x}, \frac{\mathbf{f}(\bar{\mathbf{u}}_{j+1}^n) - \mathbf{f}(\bar{\mathbf{u}}_{j-1}^n)}{2\Delta x}, \theta\,\frac{\mathbf{f}(\bar{\mathbf{u}}_{j+1}^n) - \mathbf{f}(\bar{\mathbf{u}}_j^n)}{\Delta x}\right)$$

Nessyahu-Tadmor Scheme for a Traffic Flow Model with Arrhenius Look-Ahead Dynamics

$\bar{u}_j^n \approx \dfrac{1}{\Delta x} \displaystyle\int_{C_j} u(x,t^n)\,dx$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

This solution is approximated by a piecewise linear (conservative, second-order accurate, non-oscillatory) reconstruction:

$\widetilde{u}^n(x) = \bar{u}_j^n + s_j^n(x - x_j)$ for $x \in C_j$

For example,

$$\boxed{s_j^n = \text{minmod}\left(\theta\frac{\bar{u}_j^n - \bar{u}_{j-1}^n}{\Delta x}, \frac{\bar{u}_{j+1}^n - \bar{u}_{j-1}^n}{2\Delta x}, \theta\frac{\bar{u}_{j+1}^n - \bar{u}_j^n}{\Delta x}\right)} \quad \theta \in [1,2]$$

The Nessyahu-Tadmor (NT) scheme is designed by integrating

$$u_t + F(u, U)_x = 0$$

over the space-time control volumes $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$:

$$\bar{u}^{n+1}_{j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \tilde{u}^n(x)\, dx - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \Big[F(u(x_{j+1}, t), U(x_{j+1}, t)) - F(u(x_j, t), U(x_j, t)) \Big] dt$$

Due to the finite speed of propagation, this can be reduced to:

$$\bar{u}^{n+1}_{j+\frac{1}{2}} = \frac{\bar{u}_j + \bar{u}_{j+1}}{2} + \frac{\Delta x}{8}(s^n_j - s^n_{j+1}) - \frac{\Delta t}{\Delta x}\Big[F(u^{n+\frac{1}{2}}_{j+1}, U^{n+\frac{1}{2}}_{j+1}) - F(u^{n+\frac{1}{2}}_j, U^{n+\frac{1}{2}}_j)\Big]$$

u and U at $t = t^{n+\frac{1}{2}}$ are approximated using the Taylor expansion:

$$u_j^{n+\frac{1}{2}} \approx \widetilde{u}^n(x_j) + \frac{\Delta t}{2} u_t(x_j, t^n) \qquad U_j^{n+\frac{1}{2}} \approx \widetilde{U}^n(x_j) + \frac{\Delta t}{2} U_t(x_j, t^n)$$

- $\widetilde{u}^n(x) = \bar{u}_j^n + s_j^n(x - x_j) \Longrightarrow \boxed{u_j^n := \widetilde{u}^n(x_j) = \bar{u}_j^n}$

- By integrating $\widetilde{u}^n$ from $x_{\min}$ to x we obtain the piecewise quadratic approximation of the antiderivative U:

$$\widetilde{U}^n(x) = \int_{x_{\min}}^{x} \widetilde{u}^n(\xi)\, d\xi = \Delta x \sum_{i=0}^{j-1} \bar{u}_i^n + \int_{x_{j-\frac{1}{2}}}^{x} \widetilde{u}^n(\xi)\, d\xi$$

$$= \Delta x \sum_{i=0}^{j-1} \bar{u}_i^n + \bar{u}_j^n (x - x_{j-\frac{1}{2}}) + \frac{s_j^n}{2}(x - x_{j-\frac{1}{2}})(x - x_{j+\frac{1}{2}}), \quad x \in C_j$$

In particular,

$$\boxed{U_j^n := \widetilde{U}^n(x_j) = \Delta x \sum_{i=0}^{j-1} \bar{u}_i^n + \frac{\Delta x}{2} \bar{u}_j^n - \frac{(\Delta x)^2}{8} s_j^n}$$

$$u_j^{n+\frac{1}{2}} \approx \tilde{u}^n(x_j) + \frac{\Delta t}{2} u_t(x_j, t^n) \qquad U_j^{n+\frac{1}{2}} \approx \tilde{U}^n(x_j) + \frac{\Delta t}{2} U_t(x_j, t^n)$$

$$\boxed{u_t(x_j, t^n) = -F_x(u_j^n, U_j^n)} \tag{1}$$

The space derivative F_x is computed using the (minmod) limiter:

$$F_x(u_j^n, U_j^n) = \text{minmod}\left(\theta \frac{F(\bar{u}_j^n, U_j^n) - F(\bar{u}_{j-1}^n, U_{j-1}^n)}{\Delta x}, \right.$$
$$\left. \frac{F(\bar{u}_{j+1}^n, U_{j+1}^n) - F(\bar{u}_{j-1}^n, U_{j-1}^n)}{2\Delta x}, \theta \frac{F(\bar{u}_{j+1}^n, U_{j+1}^n) - F(\bar{u}_j^n, U_j^n)}{\Delta x}\right)$$

Finally, we integrate equation (1) with respect to x to obtain:

$$\boxed{U_t(x_j, t^n) = -F(u_j^n, U_j^n)}$$

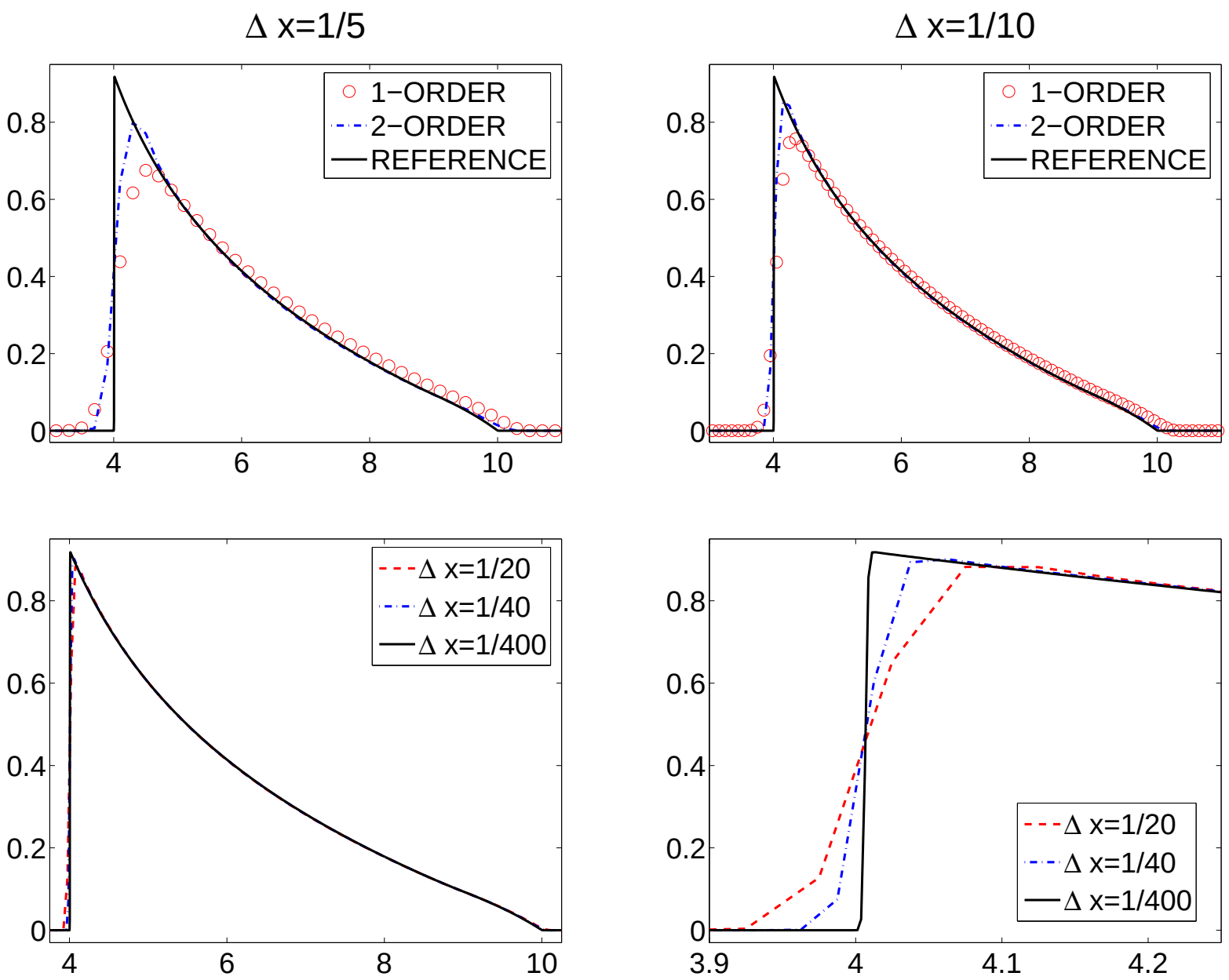
Numerical Examples

Example 1 — Red Light Traffic

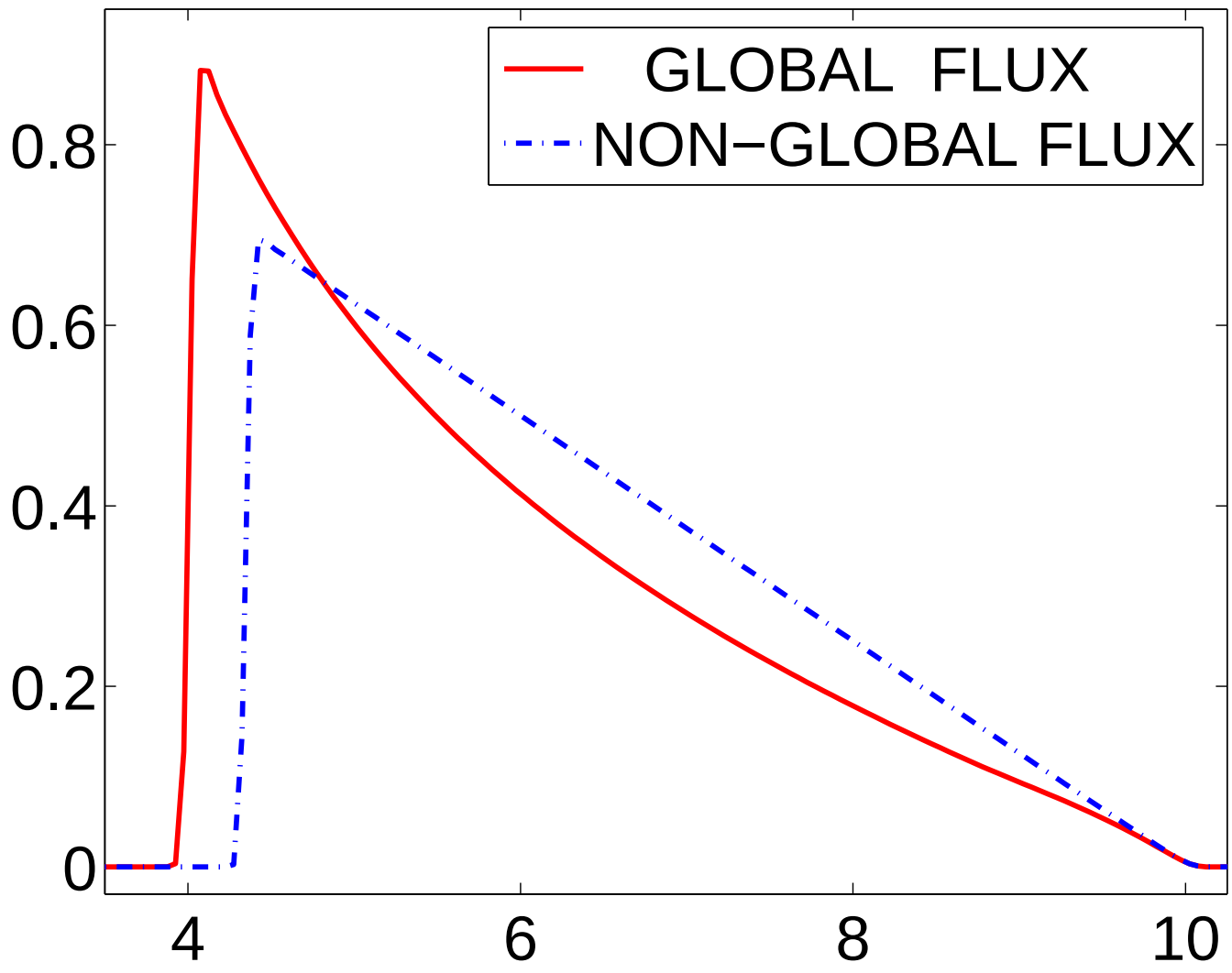
$$u(x,0) = \begin{cases} 1, & 4 < x < 6 \\ 0, & \text{otherwise} \end{cases}$$

This corresponds to a situation in which the traffic light is located at $x = 6$ and it is turned from red to green at $t = 0$.

First, fix $\gamma = 1$


Δ x=1/5
Δ x=1/10
1−ORDER
2−ORDER
REFERENCE
Δ x=1/20
Δ x=1/40
Δ x=1/400
0
0.2
0.4
0.6
0.8
4
6
8
10
3.9
4.1
4.2

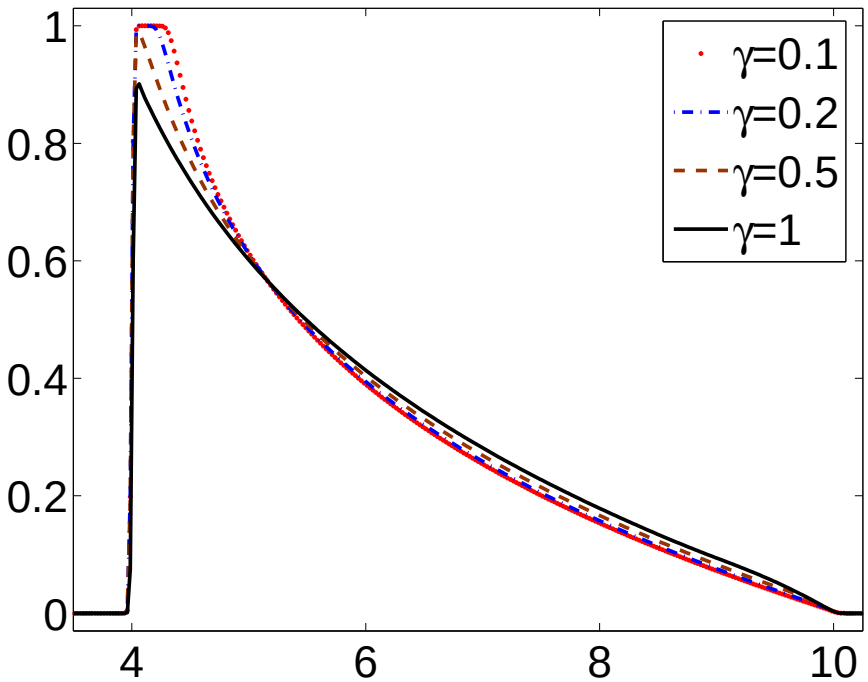
Impact of the look-ahead dynamics: global vs. non-global fluxes

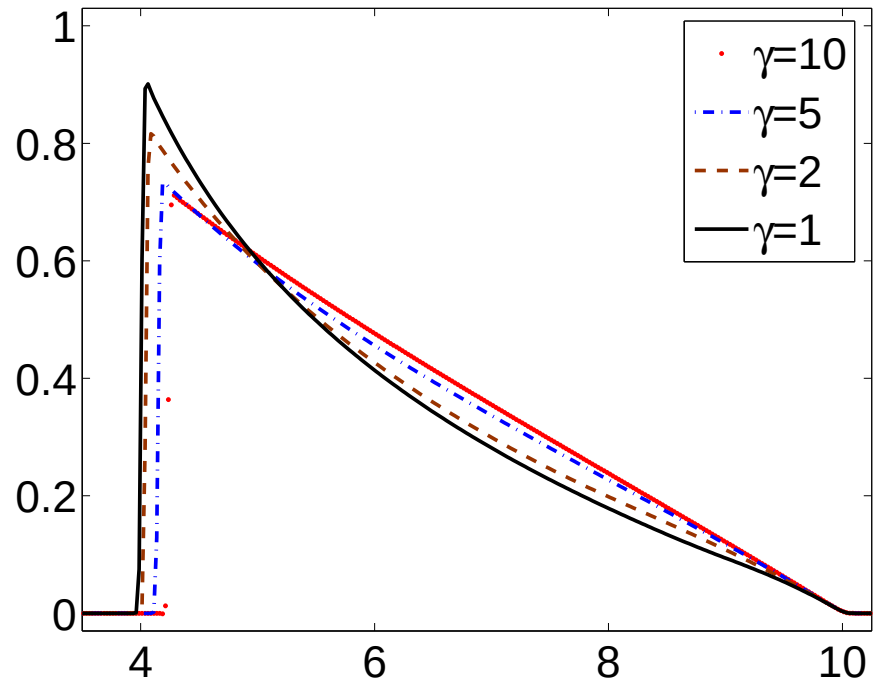


We finally demonstrate the dependence of the computed solution on the look-ahead distance γ:

- as γ decreases, the drivers can see less and the original traffic jam dissipates more slowly

- when γ is large, the effect of the global flux is small





Formal limits of the global flux as $\gamma \to \infty$ and $\gamma \to 0$

The global factor $\exp\left(-\frac{U(x+\gamma)-U(x)}{\gamma}\right) \to 1 \quad \text{as} \quad \gamma \to \infty$ assuming the total number of cars on the freeway is bounded. Therefore,

$$\boxed{F(u,U) \longrightarrow f(u) = V_{\mathrm{m}} u(1-u) \quad \text{as} \quad \gamma \to \infty}$$

that is, the global model reduces to the original, non-global one.

$\left(\dfrac{U(x+\gamma)-U(x)}{\gamma}\right) \to U_x(x) = u(x) \quad \text{as} \quad \gamma \to 0$. Therefore,

$$\boxed{F(u,U) \longrightarrow f(u)e^{-u} = V_{\mathrm{m}} u(1-u)e^{-u} \quad \text{as} \quad \gamma \to 0}$$

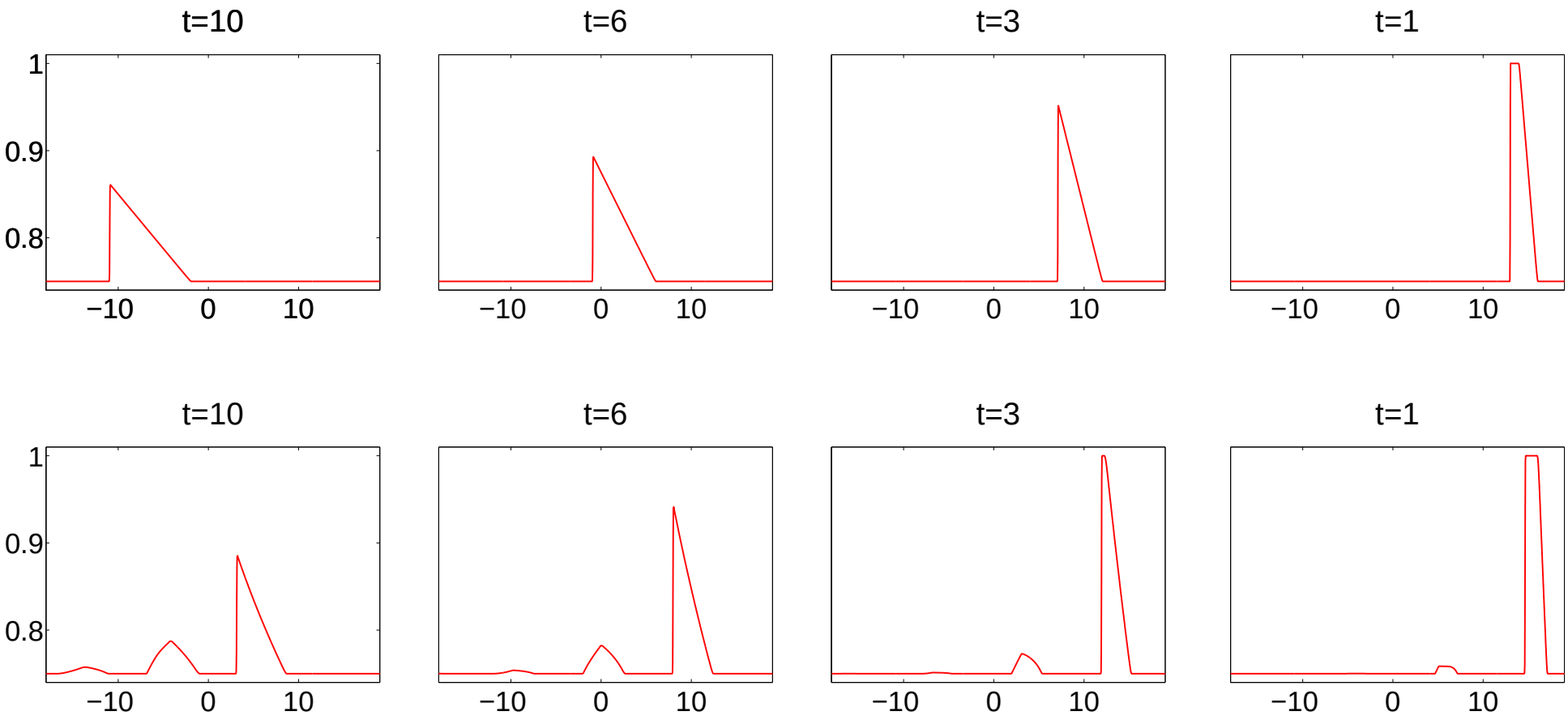
e^{-u}: "slow down" factor in the limiting low visibility case.

Example 2 — Traffic Jam on a Busy Freeway

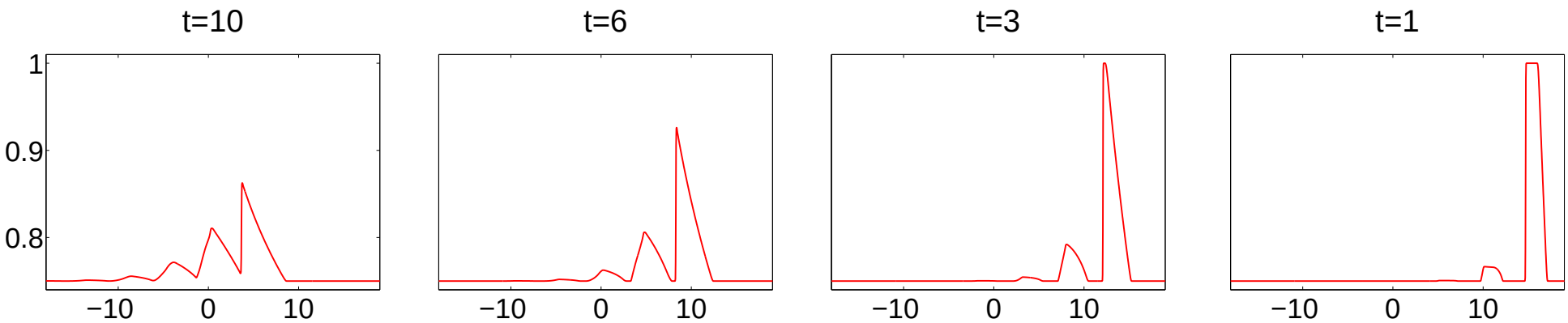
$$u(x,0) = \begin{cases} 1, & 16 < x < 18 \\ 0.75, & \text{otherwise} \end{cases}$$

Dispersive effect is much more prominent here since the waves are moving in the direction opposite to the vehicles' velocity.

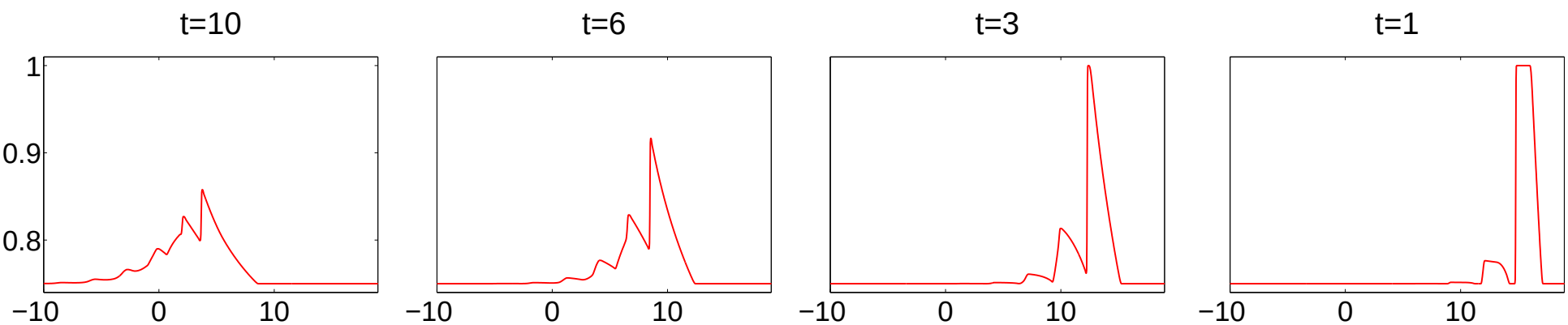
Even in the case of a large $\gamma = 10$, the global and non-global flux solutions are significantly different:



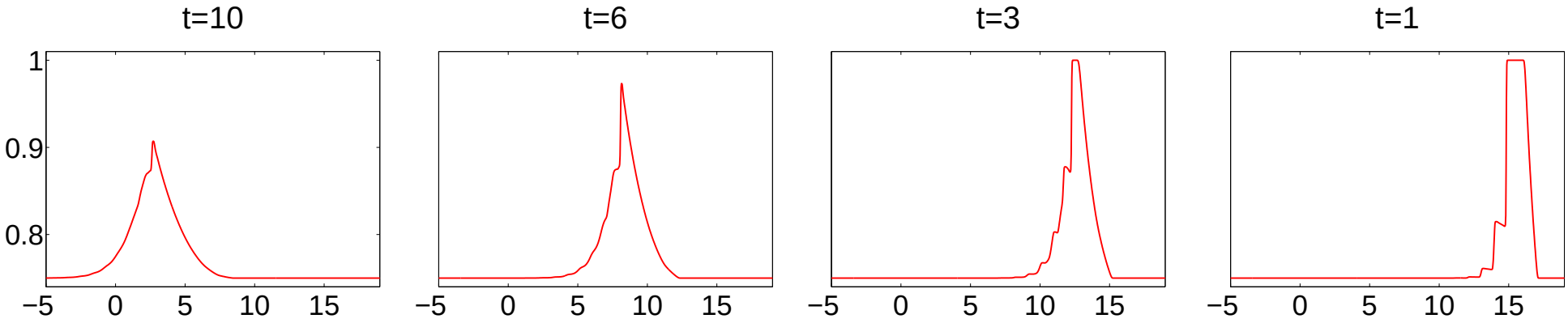
For a smaller $\gamma = 5$, the dispersive waves are getting together by time $t = 10$:



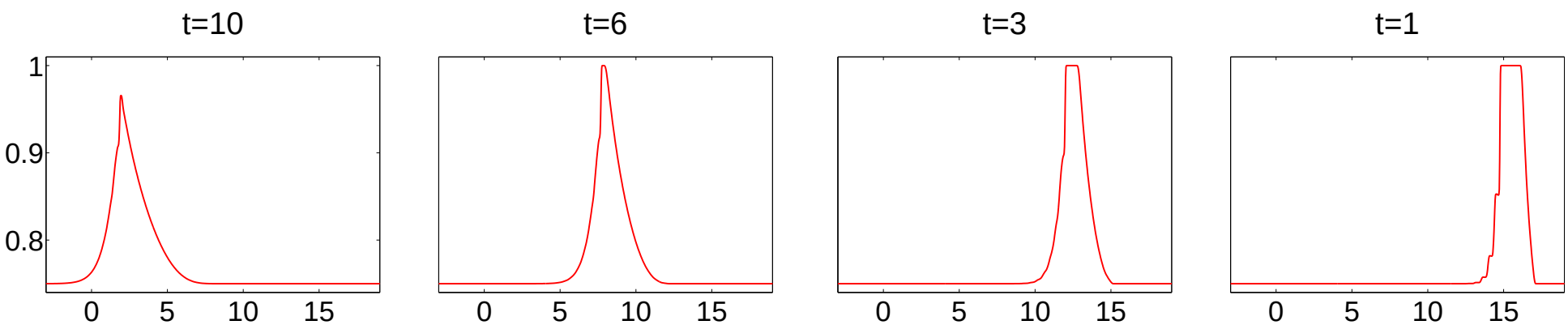
For $\gamma = 3$, the larger waves catch the smaller ones much faster now and a dispersive "wave package" is formed at about $t = 3$:



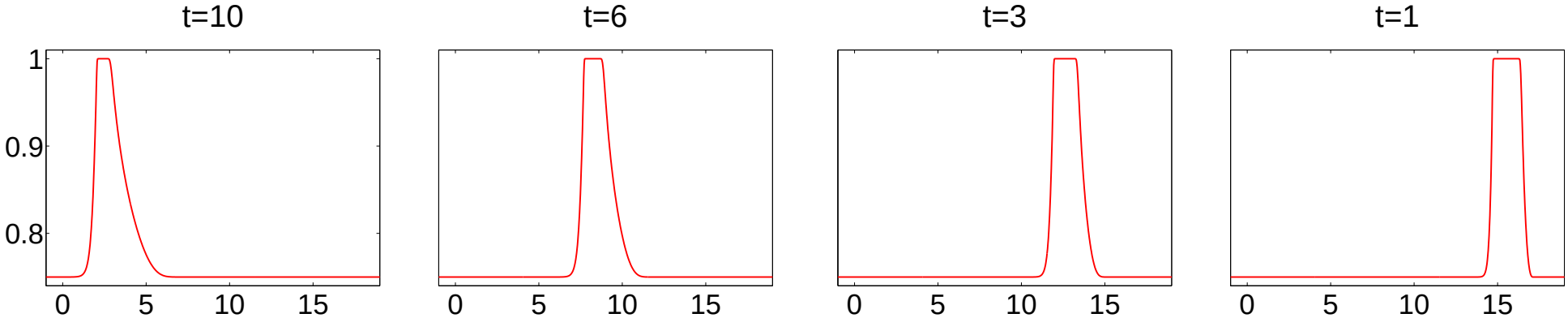
When $\gamma = 1$, a dispersive "wave package" develops right away and is later transformed into one wave:



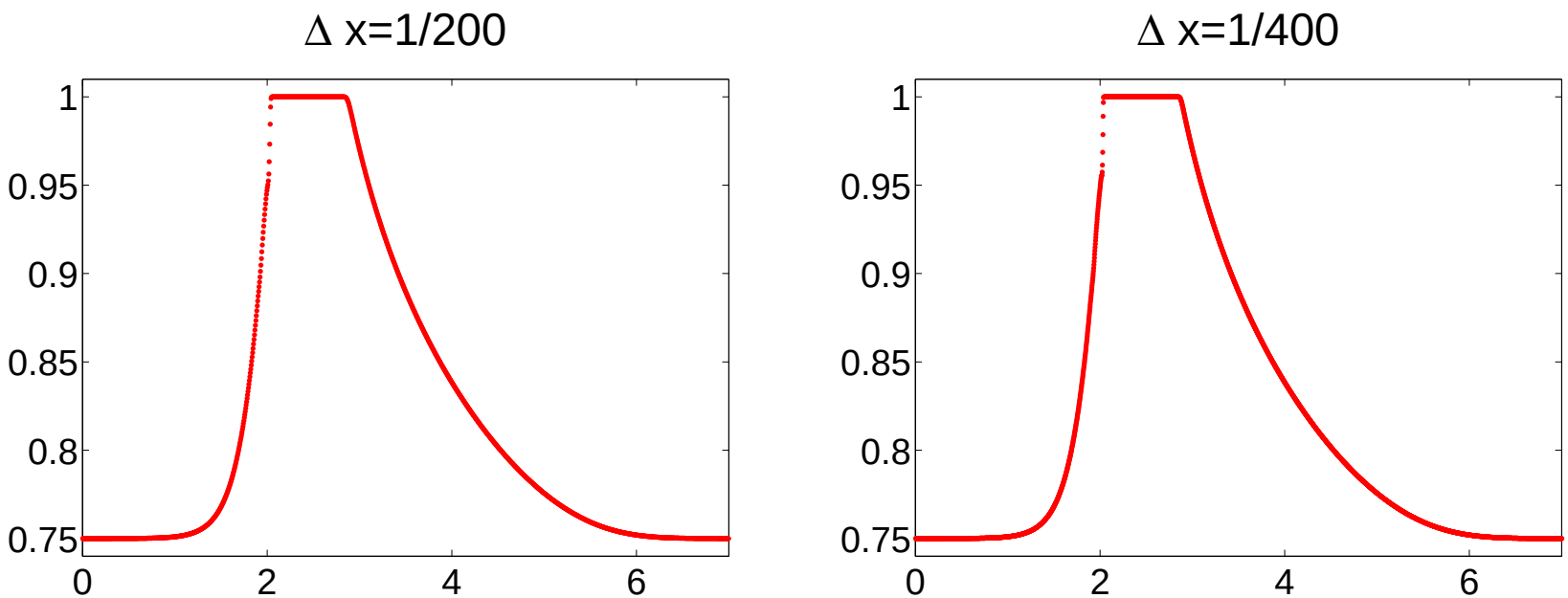
When $\gamma = 0.5$, the dispersive effect can be observed at small times only:



For even smaller $\gamma = 0.1$, no dispersive effect can be observed:



A certain smoothing effect can be observed though the solution seems to contain a sub-shock at the left edge of the wave:



- Large and intermediate values of $\gamma \Longrightarrow$ dispersive effect

- Small values of $\gamma \Longrightarrow$ smoothing effect starts dominating

The nature of these phenomena can be heuristically explained by expanding $U(x+\gamma)$ in

$$u_t + F(u,U)_x = 0, \quad F(u,U) = V_{\mathrm{m}} u(1-u) \underline{\underline{\exp\left(-\frac{U(x+\gamma)-U(x)}{\gamma}\right)}}$$

into the Taylor series about x:

$$u_t + \left[V_{\mathrm{m}}(u-u^2)\underline{\underline{e^{\mathsf{TS}}}}\right]_x = 0, \qquad \mathsf{TS} := -\left(u + \frac{\gamma}{2}u_x + \frac{\gamma^2}{6}u_{xx} + \ldots\right)$$

$$\Updownarrow$$

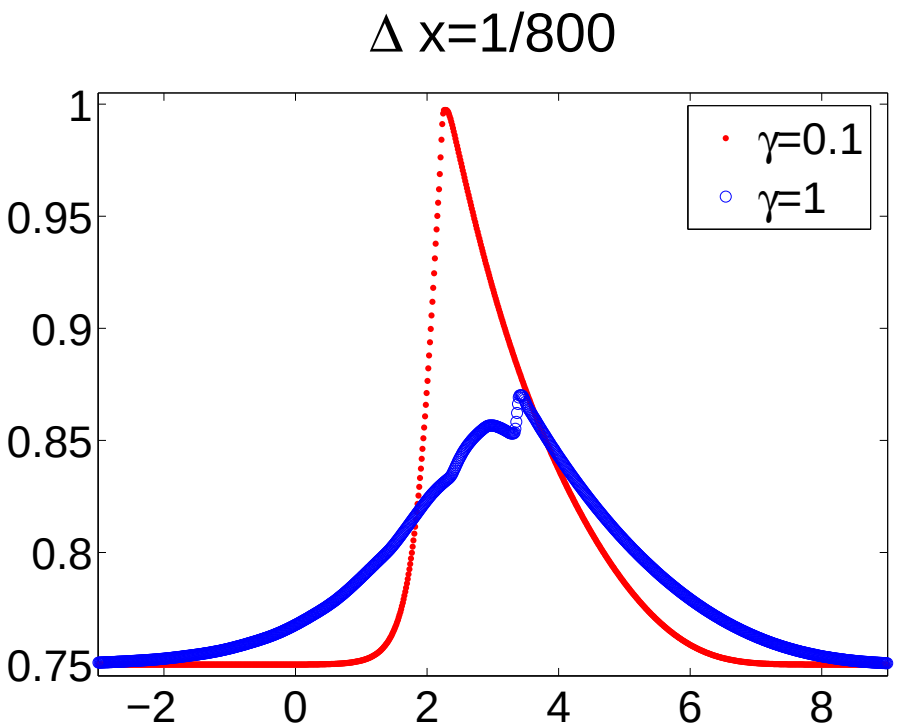
$$u_t + V_{\mathrm{m}}(1-3u+u^2)e^{\mathsf{TS}}\, u_x = V_{\mathrm{m}}(u-u^2)e^{\mathsf{TS}}\left(\frac{\gamma}{2}u_{xx} + \frac{\gamma^2}{6}u_{xxx} + \ldots\right)$$

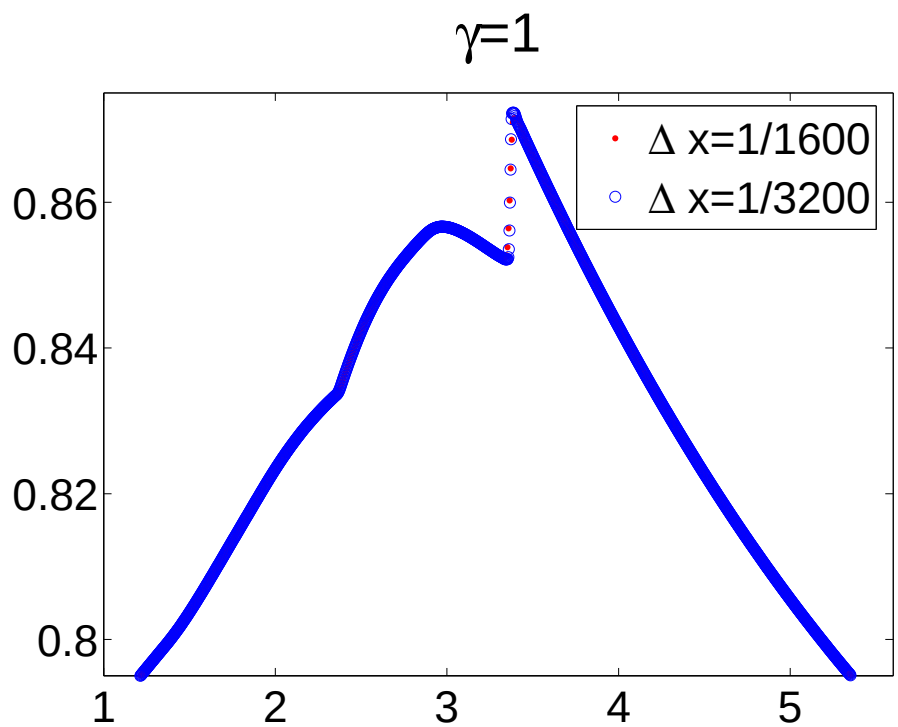
Our conjecture: for small γ initially smooth solutions may remain smooth for all t. However, the viscosity coefficient in this case is small so that smooth solutions may develop sharp gradients.

Example 3 — Numerical Breakdown Study

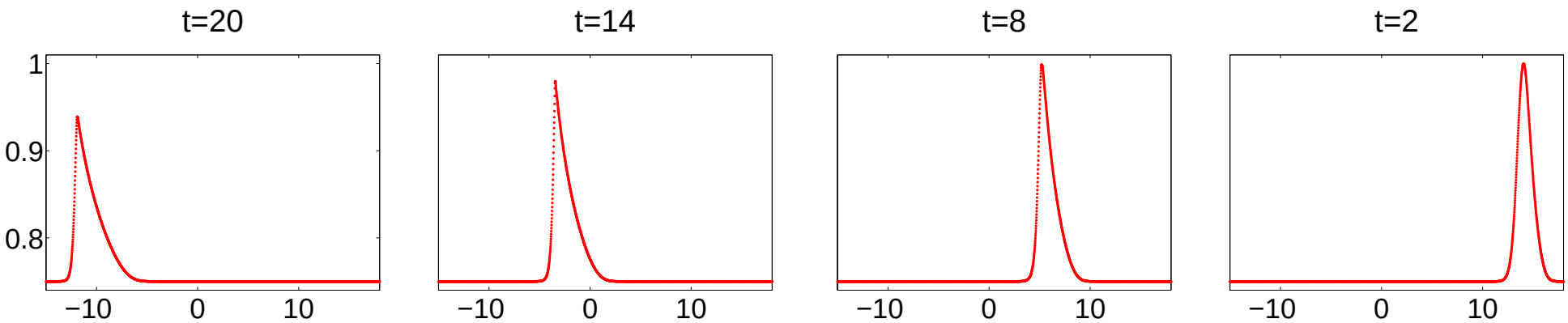
To check our conjecture, we consider the smooth initial data:

$$u(x,0) = 0.75 + 0.25e^{-(x-17)^2}$$

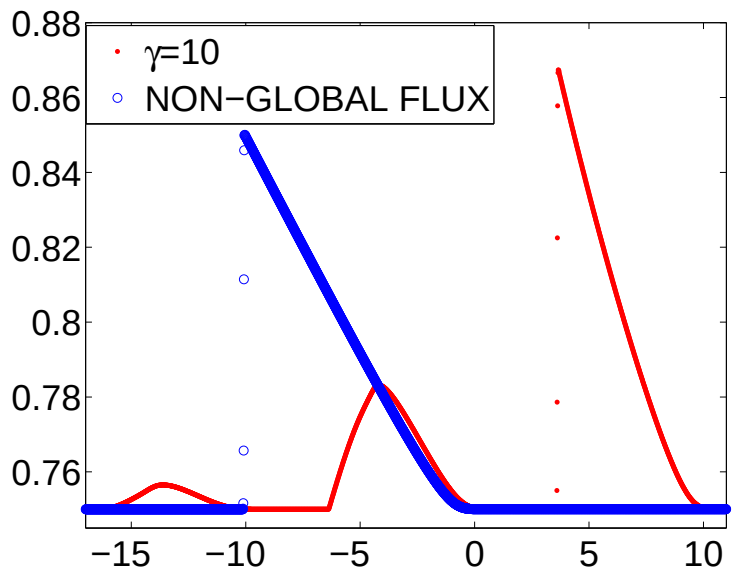




For small $\gamma = 0.1$, the solution does not seem to break down at any stage of its evolution:



For large $\gamma = 10$, the solution of the global model clearly breaks down in finite time like the solution of the non-global model:



Linear Interaction Potential

A more realistic interaction potential is a linear one:

$$J(r) = \frac{1}{\gamma}\varphi\left(\frac{r}{\gamma}\right), \qquad \varphi(r) = \begin{cases} 2-2r, & 0 < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

that is,

$$J(r) = \begin{cases} \dfrac{2}{\gamma}\left(1-\dfrac{r}{\gamma}\right), & 0 < r < \gamma \\ 0, & \text{otherwise} \end{cases}$$

In this case,

$$J \circ u(x,t) = \int_x^\infty J(y-x)u(y,t)\,dy = \frac{2}{\gamma}\int_x^{x+\gamma}\left(1-\frac{y-x}{\gamma}\right)u(y,t)\,dy.$$

or after the integration by parts,

$$J \circ u(x,t) = \frac{2}{\gamma}\left[\frac{1}{\gamma}\int_x^{x+\gamma} U(y,t)\,dy - U(x,t)\right]$$

It is easy to show that

$$J \circ u(x,t) \longrightarrow u(x,t) \quad \text{as} \quad \gamma \to 0$$

This suggests that for small γ, the modified model is not expected to be much different from the original one.

• Indeed, when the visibility is very bad, all vehicles located within the interval $(x, x+\gamma]$ are expected to have almost the same influence on the driver of the car located at x.

• However, when γ is not too small, the use of the linear interaction potential is motivated by the fact that the vehicles located further away from x (but still visible to the driver there) typically have smaller influence on the driver's behavior then the vehicles that are nearby.

Formula

$$J \circ u(x,t) = \frac{2}{\gamma}\left[\frac{1}{\gamma}\int_{x}^{x+\gamma} U(y,t)\,dy - U(x,t)\right]$$

can be rewritten as

$$J \circ u(x,t) = \frac{2}{\gamma}\left[\frac{\mathcal{U}(x+\gamma,t) - \mathcal{U}(x,t)}{\gamma} - U(x,t)\right]$$

where $\mathcal{U}$ denotes the antiderivative of U:

$$\mathcal{U}(x,t) := \int_{-\infty}^{x} U(\xi,t)\,d\xi$$

The modified model is:

$$u_t + F(u, U, \mathcal{U})_x = 0$$

$$F(u, U, \mathcal{U}) = V_{\mathsf{m}} u(1-u) \underline{\underline{\exp\left(-\frac{2}{\gamma}\left[\frac{\mathcal{U}(x+\gamma, t) - \mathcal{U}(x, t)}{\gamma} - U(x, t)\right]\right)}}$$

Compare with

$$u_t + F(u, U)_x = 0$$

$$F(u, U) = V_{\mathsf{m}} u(1-u) \underline{\underline{\exp\left(-\frac{U(x+\gamma) - U(x)}{\gamma}\right)}}$$

Nessyahu-Tadmor Scheme for a Traffic Flow Model with Linear Interaction Potential

$$u_t + F(u, U, \mathcal{U})_x = 0$$

$$\begin{aligned}\bar{u}^{n+1}_{j+\frac{1}{2}} &= \frac{\bar{u}_j + \bar{u}_{j+1}}{2} + \frac{\Delta x}{8}(s^n_j - s^n_{j+1}) \\ &\quad -\lambda \left[F(u, U, \mathcal{U})\Big|_{(x_{j+1}, t^{n+\frac{1}{2}})} - F(u, U, \mathcal{U})\Big|_{(x_j, t^{n+\frac{1}{2}})} \right]\end{aligned}$$

Therefore, to complete the description of the NT scheme we need to provide a recipe for the calculation of the intermediate point values $\{\mathcal{U}(x_j, t^{n+\frac{1}{2}})\}$.

From the Taylor expansion

$$\mathcal{U}(x_j, t^{n+\frac{1}{2}}) \approx \mathcal{U}(x_j, t^n) + \frac{\Delta t}{2}\mathcal{U}_t(x_j, t^n)$$

The approximate values of $\{\mathcal{U}(x_j, t^n)\}$ can be obtained by integrating the piecewise quadratic function

$$\widetilde{U}^n = \Delta x \sum_{i=0}^{j-1} \bar{u}_i^n + \bar{u}_j^n (x - x_{j-\frac{1}{2}}) + \frac{s_j^n}{2}(x - x_{j-\frac{1}{2}})(x - x_{j+\frac{1}{2}}), \;\; x \in C_j$$

which results in a piecewise cubic approximation of $\mathcal{U}(x, t^n)$:

$$\widetilde{\mathcal{U}}^n(x) = \int_{x_{\min}}^{x} \widetilde{U}^n(\xi)\, d\xi = \Delta x \sum_{i=0}^{j-1} w_i^n + w_j^n (x - x_{j-\frac{1}{2}})$$
$$+ \frac{\bar{u}_j^n}{2}(x - x_{j-\frac{1}{2}})(x - x_{j+\frac{1}{2}}) + \frac{s_j^n}{6}(x - x_{j-\frac{1}{2}})(x - x_j)(x - x_{j+\frac{1}{2}}), \;\; x \in C_j$$

Finally, integrating equation

$$U_t(x_j, t^n) = -F(u_j^n, U_j^n)$$

we obtain

$$\mathcal{U}_t(x_j, t^n) = -\int_{x_{\text{min}}}^{x} F(u(\xi, t^n), U(\xi, t^n), \mathcal{U}(\xi, t^n))\, d\xi$$

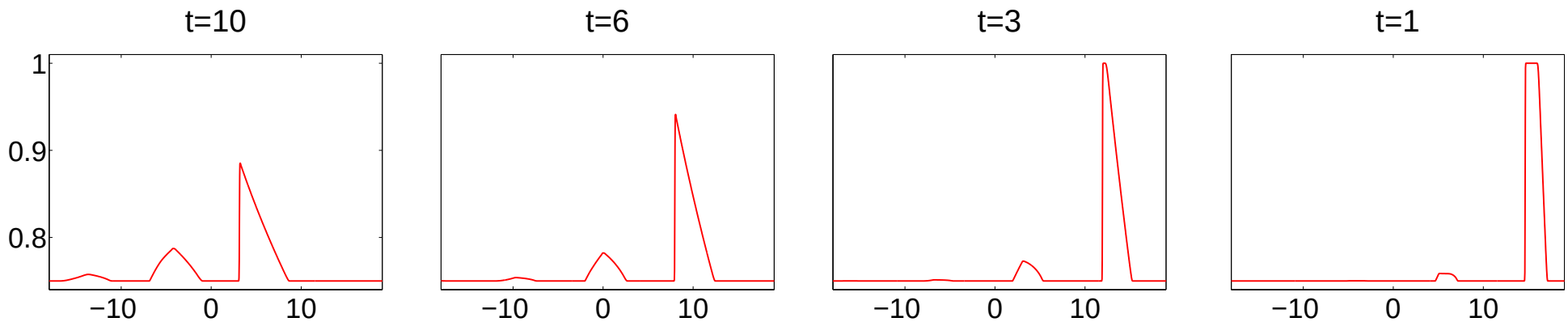
which can be calculated by exact integration of the piecewise linear approximation of F.

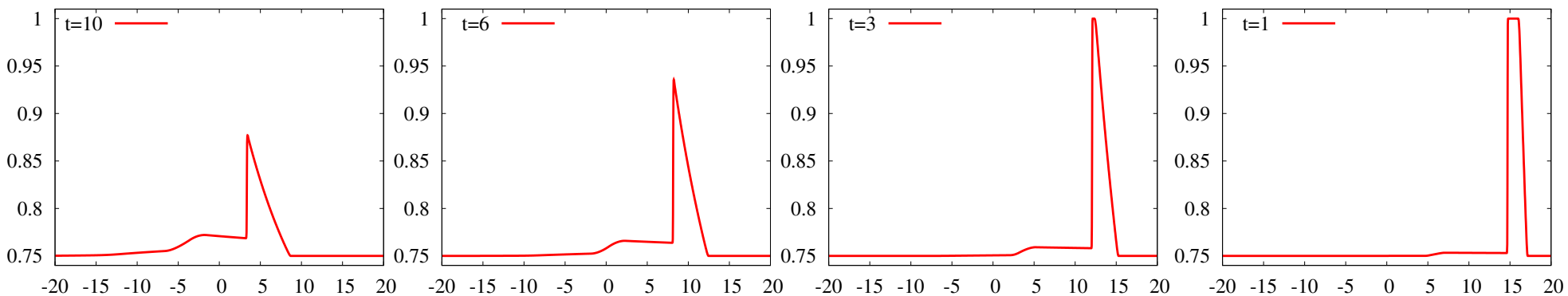
Back to Example 2 — Traffic Jam on a Busy Freeway

$$u(x,0) = \begin{cases} 1, & 16 < x < 18 \\ 0.75, & \text{otherwise} \end{cases}$$

$\gamma = 10$

Constant interaction potential

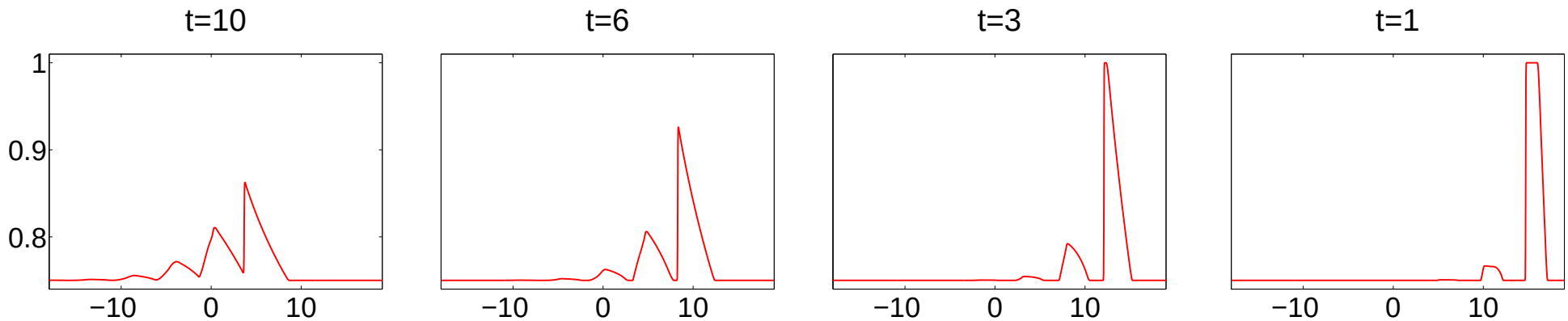


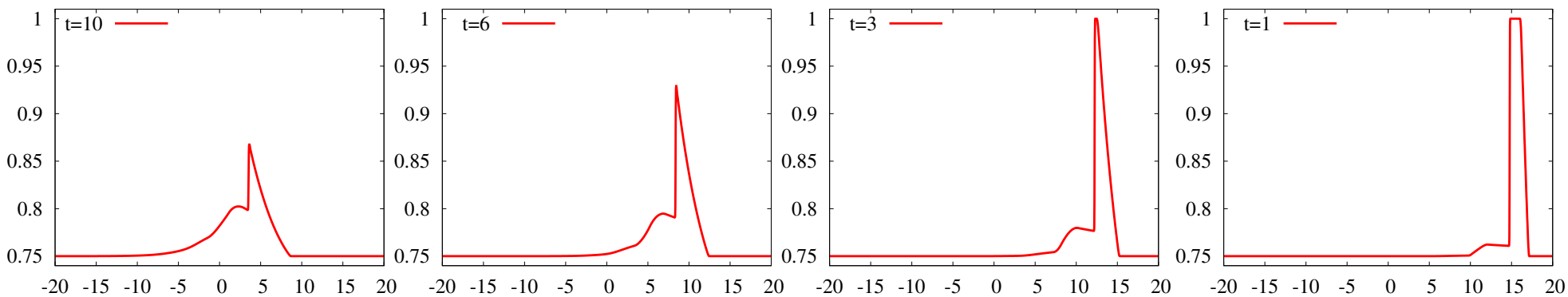


Linear interaction potential

$\gamma = 5$

Constant interaction potential

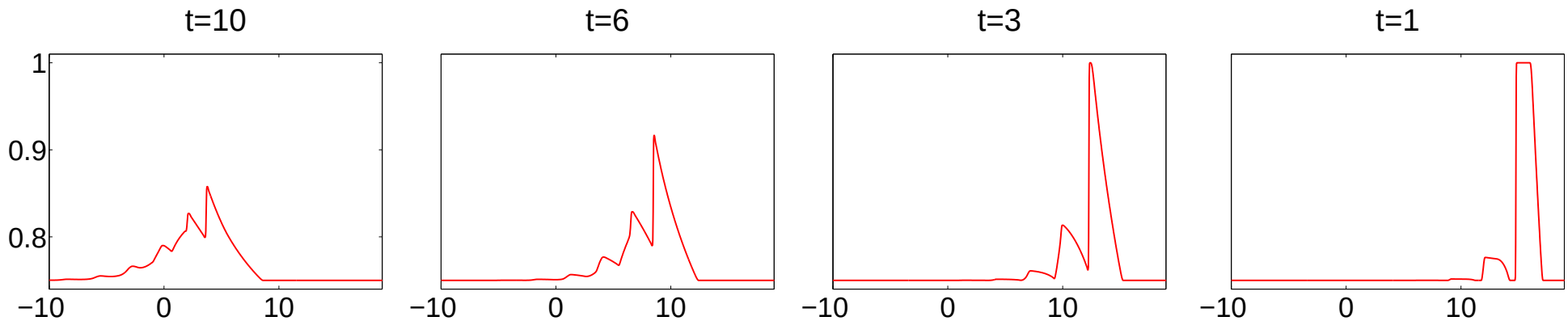


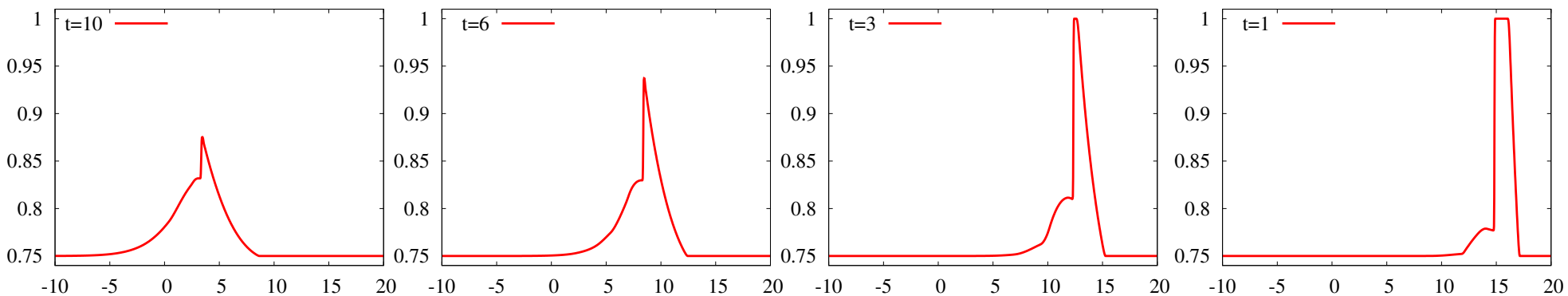


Linear interaction potential

$\gamma = 3$

Constant interaction potential

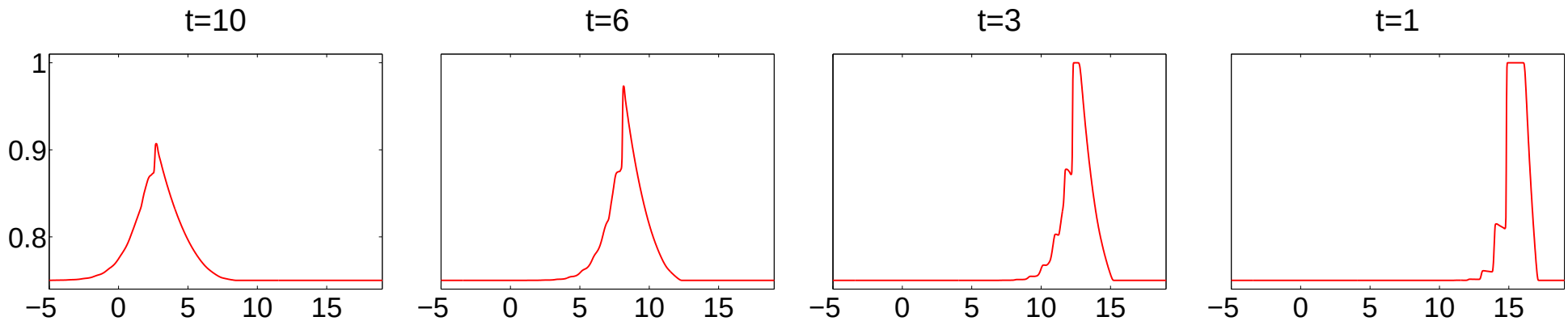


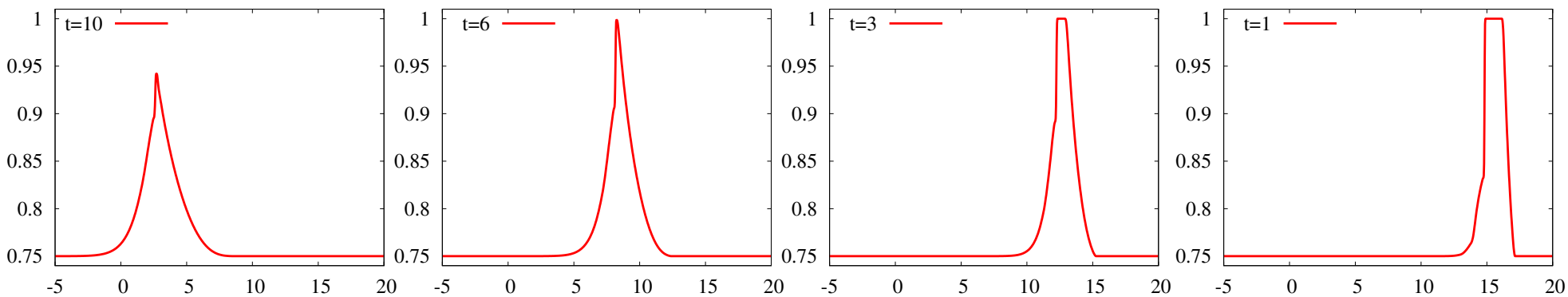


Linear interaction potential

$\gamma = 1$

Constant interaction potential

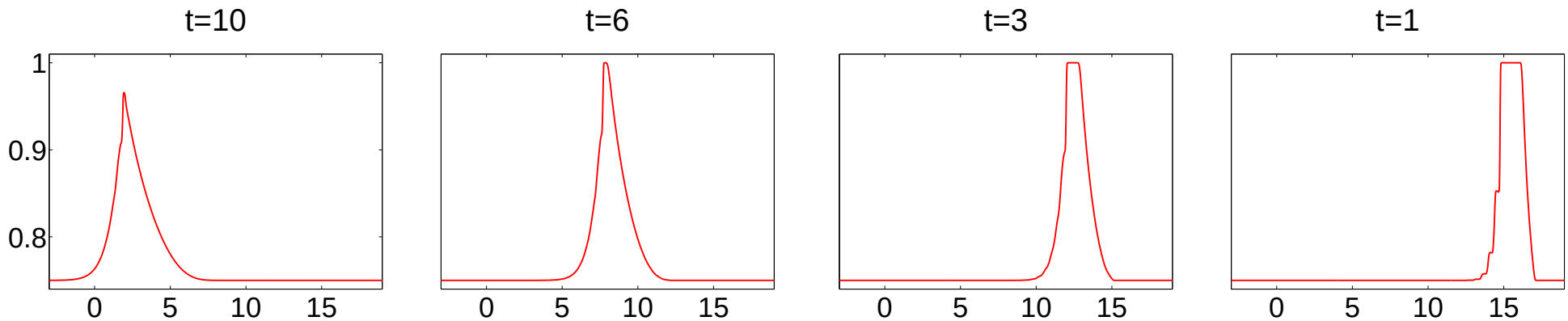


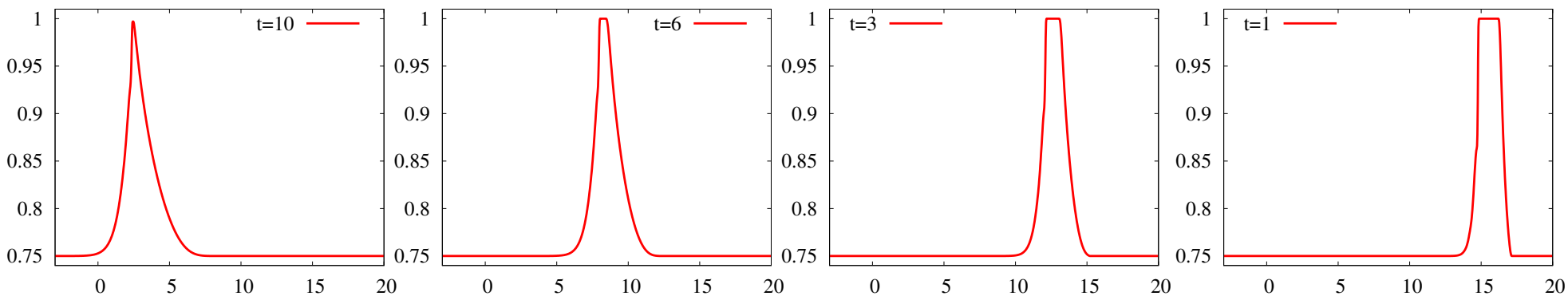


Linear interaction potential

$\gamma = 0.5$

Constant interaction potential

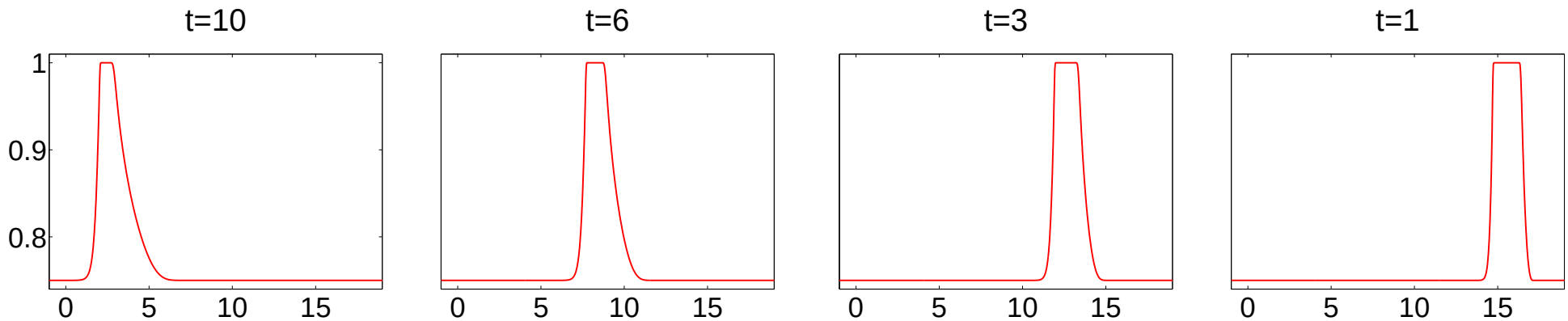


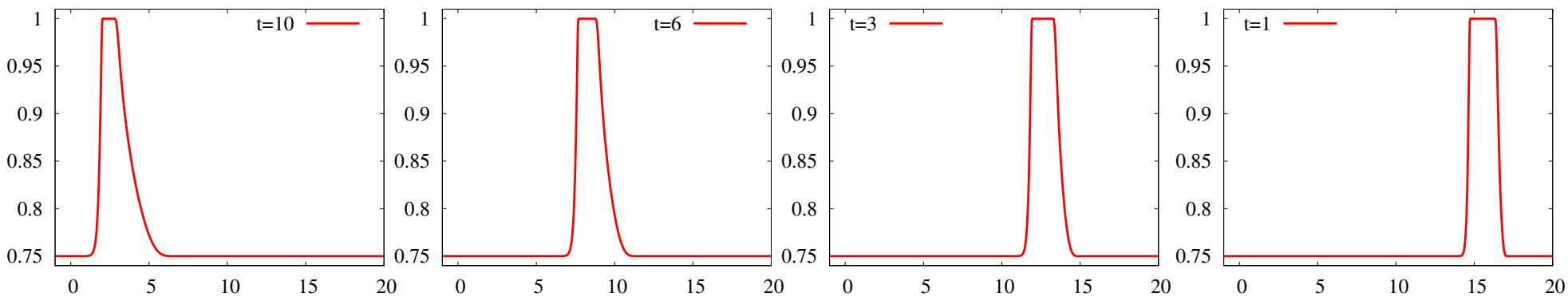


Linear interaction potential

$\gamma = 0.1$

Constant interaction potential





Linear interaction potential

Models of Formation and Evolution of Traffic Jams

[H.J. Payne; 1971,1979] $\rightarrow$ first "second-order" fluid dynamics models of traffic flow were proposed

[C. Daganzo; 1995] $\rightarrow$ These models admit absurd results:

- negative car density
- backward car movement

since cars in traffic have properties usual fluids do not have

[A. Aw, M. Rascle; 2000, 2002] $\rightarrow$ A new "second-order" model was derived from the microscopic Follow-the-Leader model

Aw-Rascle Model

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho w)_t + (\rho w u)_x = 0, \quad w = u + p(\rho) \end{cases}$$

$\rho(x,t)$: a relative car density $(0 \le \rho \le 1)$

$u > 0$: dimensionless velocity

w: drivers "preferred velocity"

$p(\rho)$: velocity offset $(0 \le p(\rho) \le \infty)$

Example ([A. Aw, M. Rascle; 2000, 2002])

$$\boxed{p(\rho) = \rho^\gamma}$$

Not the best choice since the region $[0, \rho^*] \times [0, u^*]$ is not invariant

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho w)_t + (\rho w u)_x = 0, \quad w = u + p(\rho) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ (\partial_t + u\partial_x)(u + p(\rho)) = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{u^2}{2}\right)_x = \rho p'(\rho) u_x \end{cases}$$

Compare with the pressureless gas dynamics (PGD) system:

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{u^2}{2}\right)_x = 0 \end{cases} \Longleftrightarrow \begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2)_x = 0 \end{cases}$$

Example [F. Berthelin, P. Degond, M. Delitala, M. Rascle; 2008]

$$p(\rho) = \left(\frac{1}{\rho} - \frac{1}{\rho^*}\right)^{-\gamma} \quad \text{for} \quad \rho \leq \rho^*, \quad \gamma > 0$$

ρ^*: maximal density

Notice that

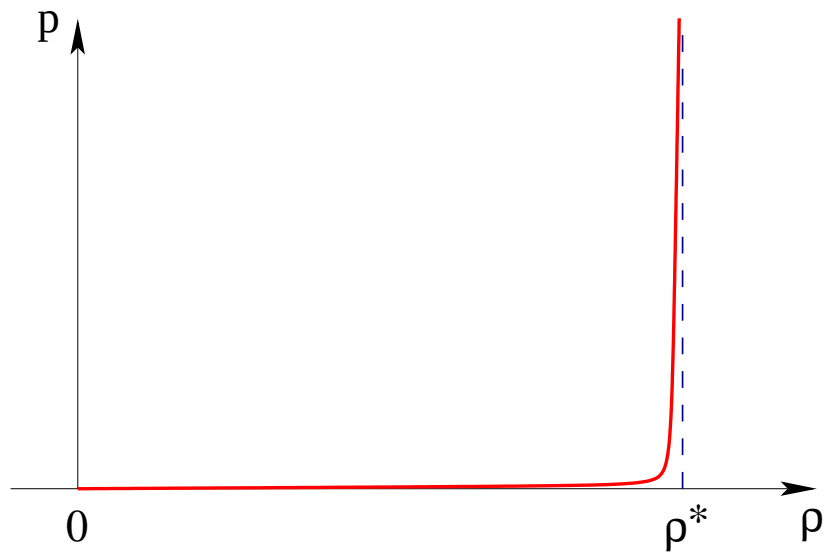
$$p(\rho) \to \infty \quad \text{as} \quad \rho \to \rho^*$$

For the modified Aw-Rascle model the region $[0, \rho^*] \times [0, u^*]$ is invariant

Modeling Traffic Jams

[F. Berthelin, P. Degond, M. Delitala, M. Rascle; 2008]

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = 0 \\ (\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(\rho^\varepsilon)) = 0, \quad p(\rho^\varepsilon) = \left(\frac{1}{\rho^\varepsilon} - \frac{1}{\rho^*}\right)^{-\gamma} \end{cases}$$



! No velocity offset unless $\rho \sim \rho^*$

! This is a very stiff problem for small ε

$\varepsilon \longrightarrow 0$

For a nonsingular $p(\rho)$ (e.g., $p(\rho) = \rho^\gamma$), $\varepsilon p(\rho^\varepsilon) \to 0$ and

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = 0 \\ (\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(\rho^\varepsilon)) = 0 \end{cases} \longrightarrow \begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left(\frac{u^2}{2}\right)_x = 0 \end{cases} \quad (PGD)$$

For $p(\rho) = \left(\dfrac{1}{\rho} - \dfrac{1}{\rho^*}\right)^{-\gamma}$, $\varepsilon p(\rho^\varepsilon) \to \widehat{p}(x,t) \geq 0$

! $\widehat{p}(x,t)$ may become positive and finite

The formal limit of the Rescaled Modified Aw-Rascle Model is the Constrained PGD system:

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = 0 \\ (\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(\rho^\varepsilon)) = 0 \end{cases} \longrightarrow \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ (\partial_t + u\partial_x)(u + \widehat{p}) = 0 \\ 0 \leq \rho \leq \rho^*, \ \widehat{p} \geq 0, \ (\rho^* - \rho)\widehat{p} = 0 \end{cases}$$

Constrained PGD System

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ (\partial_t + u\partial_x)(u + \widehat{p}) = 0 \\ 0 \leq \rho \leq \rho^*, \ \widehat{p} \geq 0, \ (\rho^* - \rho)\widehat{p} = 0 \end{cases}$$

! This system is still ill posed since the velocity offset $\widehat{p}$ is undefined inside the clusters, that is, in the regions $\rho = \rho^*$

! Rigorous passage to the $\varepsilon \to 0$ limit shows that

- $u_x = 0$ inside the clusters
- When two cluster collide, they form a new cluster moving with the velocity of the cluster on the right.

Numerical Methods for the Constrained PGD System

Method 1: **Finite-Volume Upwind Scheme**

The idea: at each time step solve the PGD system and correct the velocities at the edges of clusters

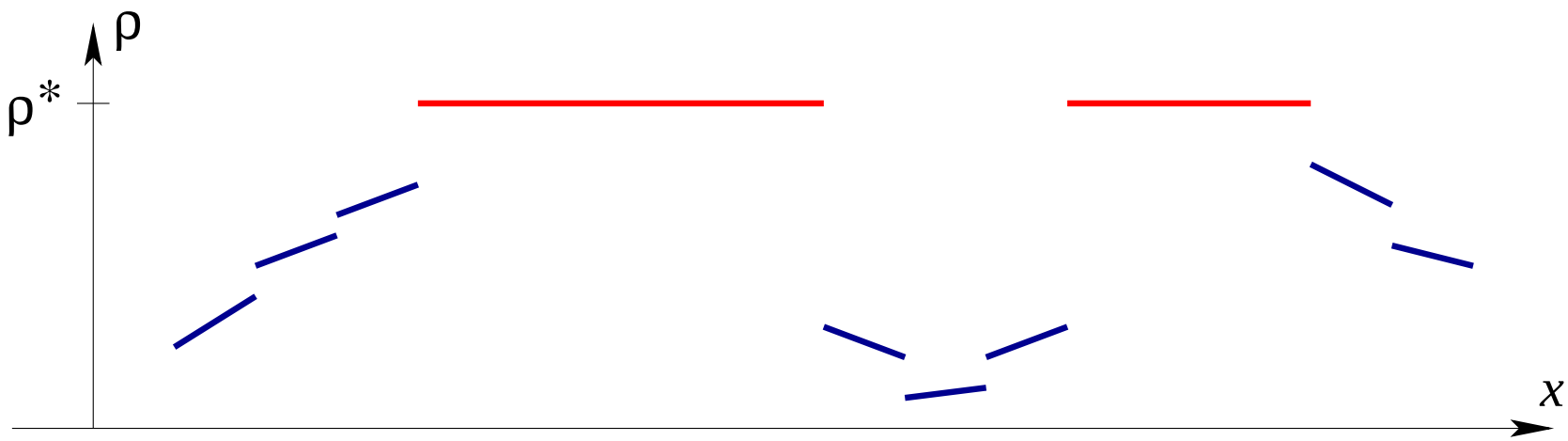
Assume that at some time level t the computed solution $\{\overline{\rho}_j(t), \overline{u}_j(t)\}$ is available. Then it is evolved to $t + \Delta t$:

$$\left\{ \begin{array}{c} \overline{\rho}_j(t) \\ \overline{u}_j(t) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \widetilde{\rho}^n(\cdot, t) \\ \widetilde{u}^n(\cdot, t) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} H^{\rho}_{j+\frac{1}{2}}(t) \\ H^{u}_{j+\frac{1}{2}}(t) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \overline{\rho}_j(t + \Delta t) \\ \overline{u}_j(t + \Delta t) \end{array} \right\}$$

$$\frac{d}{dt} \left(\begin{array}{c} \overline{\rho}_j(t) \\ \overline{u}_j(t) \end{array} \right) = - \frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x}, \qquad \mathbf{H}_{j+\frac{1}{2}}(t) := \left(\begin{array}{c} H^{\rho}_{j+\frac{1}{2}}(t) \\ H^{u}_{j+\frac{1}{2}}(t) \end{array} \right)$$

To incorporate the constraint, a special treatment of clusters is proposed.

First, clusters are marked using I_j



I_j		0	0	0	1	1	1	1	1	0	0	0	1	1	1	0	0	

Piecewise Linear Reconstruction

$$\begin{aligned}\tilde{\rho}(t) &= \overline{\rho}_j(t) + (\rho_x)_j(x - x_j)\\ \tilde{u}(t) &= \overline{u}_j(t) + (u_x)_j(x - x_j)\end{aligned} \quad \text{for } x \in C_j$$

Since the eigenvalues of the PGD are equal to u, they are nonnegative and thus we only need to compute the left-sided point values at each cell interface:

$$\boxed{\begin{aligned}\rho^-_{j+\frac{1}{2}}(t) &:= \overline{\rho}_j(t) + \frac{\Delta x}{2}(\rho_x)_j\\ u^-_{j+\frac{1}{2}}(t) &:= \overline{u}_j(t) + \frac{\Delta x}{2}(u_x)_j\end{aligned}}$$

Velocity Correction <u>behind</u> the Cluster:

`If` $I_j(t) = 0$ `and` $I_{j+1}(t) = 1$ `and`

$\rho^-_{j+\frac{1}{2}}(t)u^-_{j+\frac{1}{2}}(t) > \rho^-_{j+\frac{3}{2}}(t)u^-_{j+\frac{3}{2}}(t)$ `then` correct the velocity

$$u^-_{j+\frac{1}{2}}(t) := \frac{\rho^-_{j+\frac{3}{2}}(t)u^-_{j+\frac{3}{2}}(t)}{\rho^-_{j+\frac{1}{2}}(t)}$$

CFL Condition

$$\frac{d}{dt}\overline{\rho}_j(t) = -\frac{H^{\rho}_{j+\frac{1}{2}}(t) - H^{\rho}_{j-\frac{1}{2}}(t)}{\Delta x}, \qquad H^{\rho}_{j+\frac{1}{2}}(t) = \rho^{-}_{j+\frac{1}{2}}(t)u^{-}_{j+\frac{1}{2}}(t)$$

For the Forward Euler method,

$$\overline{\rho}^{n+1}_j = \overline{\rho}^n_j - \lambda\left(\rho^{-}_{j+\frac{1}{2}}u^{-}_{j+\frac{1}{2}} - \rho^{-}_{j-\frac{1}{2}}u^{-}_{j-\frac{1}{2}}\right) \leq \rho^*$$

provided either

$$\rho^{-}_{j+\frac{1}{2}}u^{-}_{j+\frac{1}{2}} - \rho^{-}_{j-\frac{1}{2}}u^{-}_{j-\frac{1}{2}} \geq 0$$

or

$$\lambda \leq \frac{\rho^* - \rho^n_j}{\rho^{-}_{j-\frac{1}{2}}u^{-}_{j-\frac{1}{2}} - \rho^{-}_{j+\frac{1}{2}}u^{-}_{j+\frac{1}{2}}}$$

Question: Will the time steps be too small?

Cluster Velocity Correction

After the evolution step, the new solution $\{\overline{\rho}_j(t+\Delta t), \overline{u}_j(t+\Delta t)\}$ and cluster markers $\{I_j(t+\Delta t)\}$ are available

Assume that $I_{j-q} = 0,\ I_{j-q+1} = \ldots = I_j = 1,\ I_{j+1} = 0$. Then

`If` $\overline{u}_j(t+\Delta t) > \overline{u}_{j+1}(t+\Delta t)$ `and` $\overline{\rho}_{j+1}(t+\Delta t) > 0$ `set`

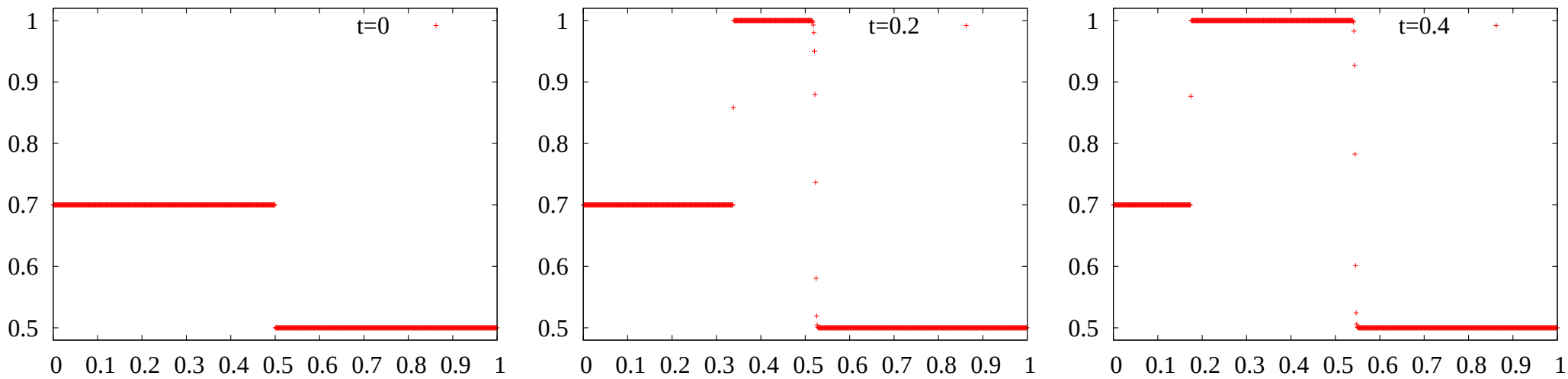
$$\overline{u}_j(t+\Delta t) := \overline{u}_{j+1}(t+\Delta t)$$

`If` $\overline{u}_j(t+\Delta t) > \overline{u}_{j+1}(t+\Delta t)$ `and` $\overline{\rho}_{j+1}(t+\Delta t) = 0$ `set`

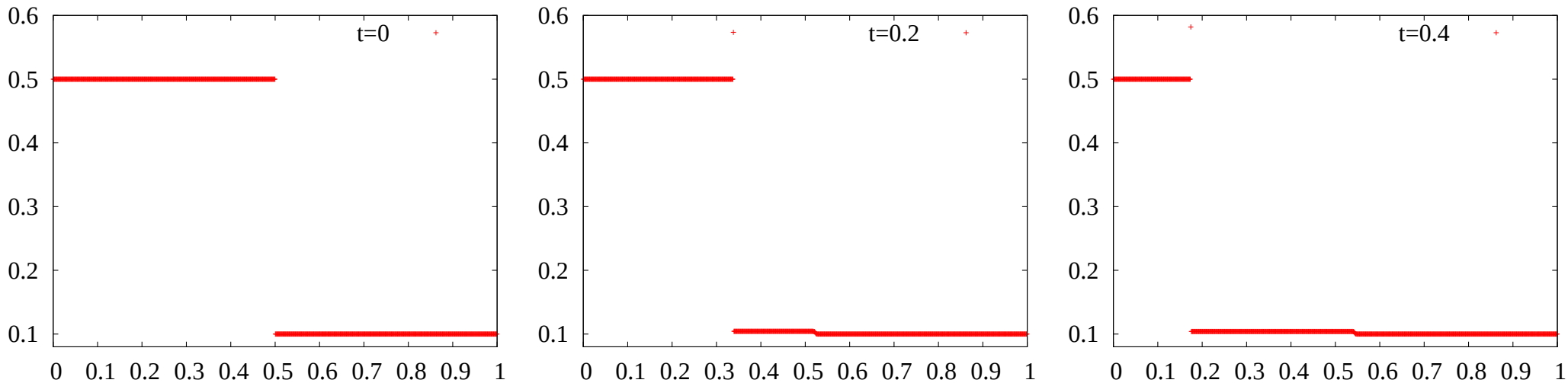
$$\overline{u}_j(t+\Delta t) := w$$

w: preferred velocity

Example 1 — Riemann Initial Data

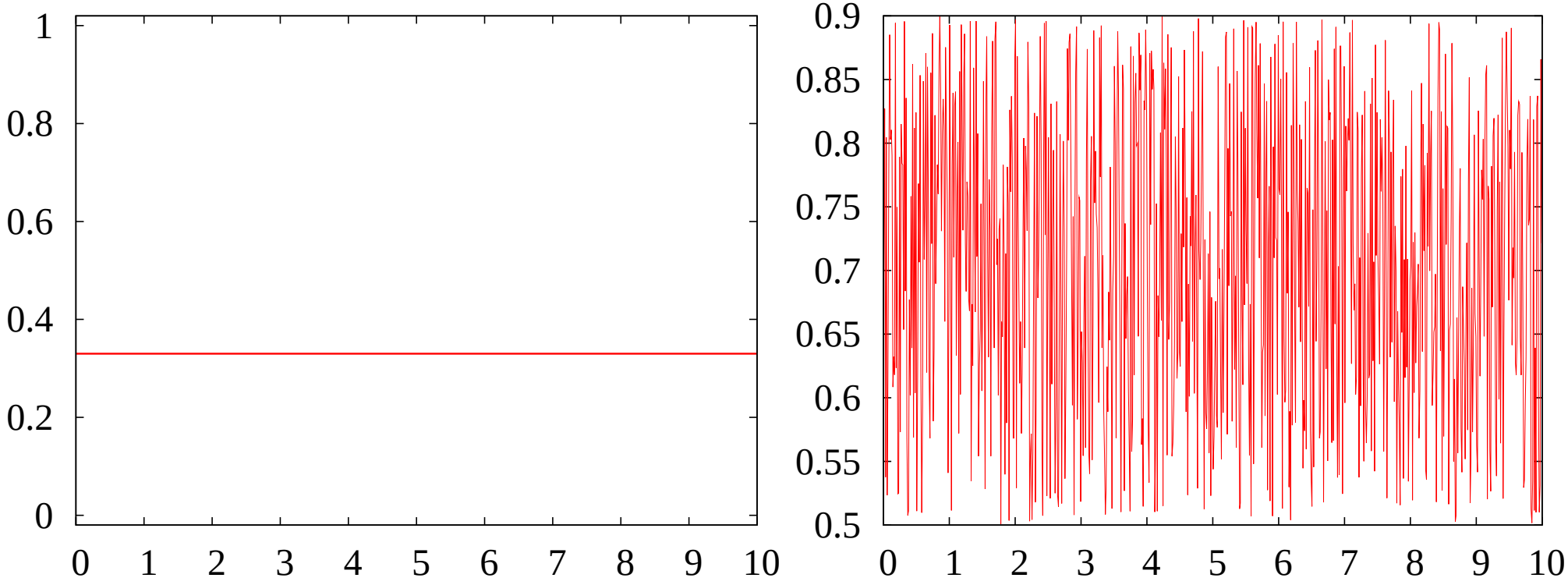


Density

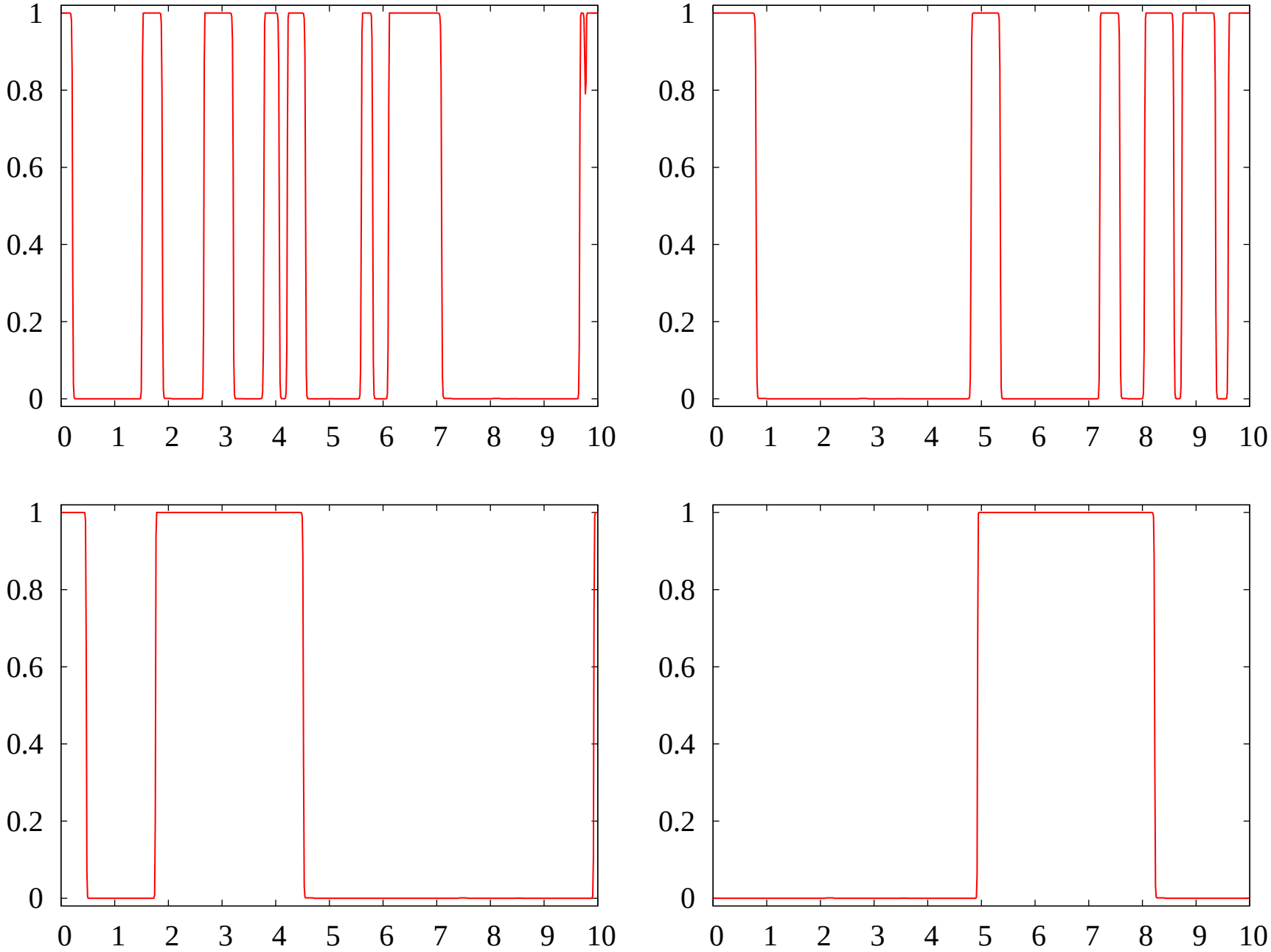


Velocity

Example 2 — Random Initial Velocity



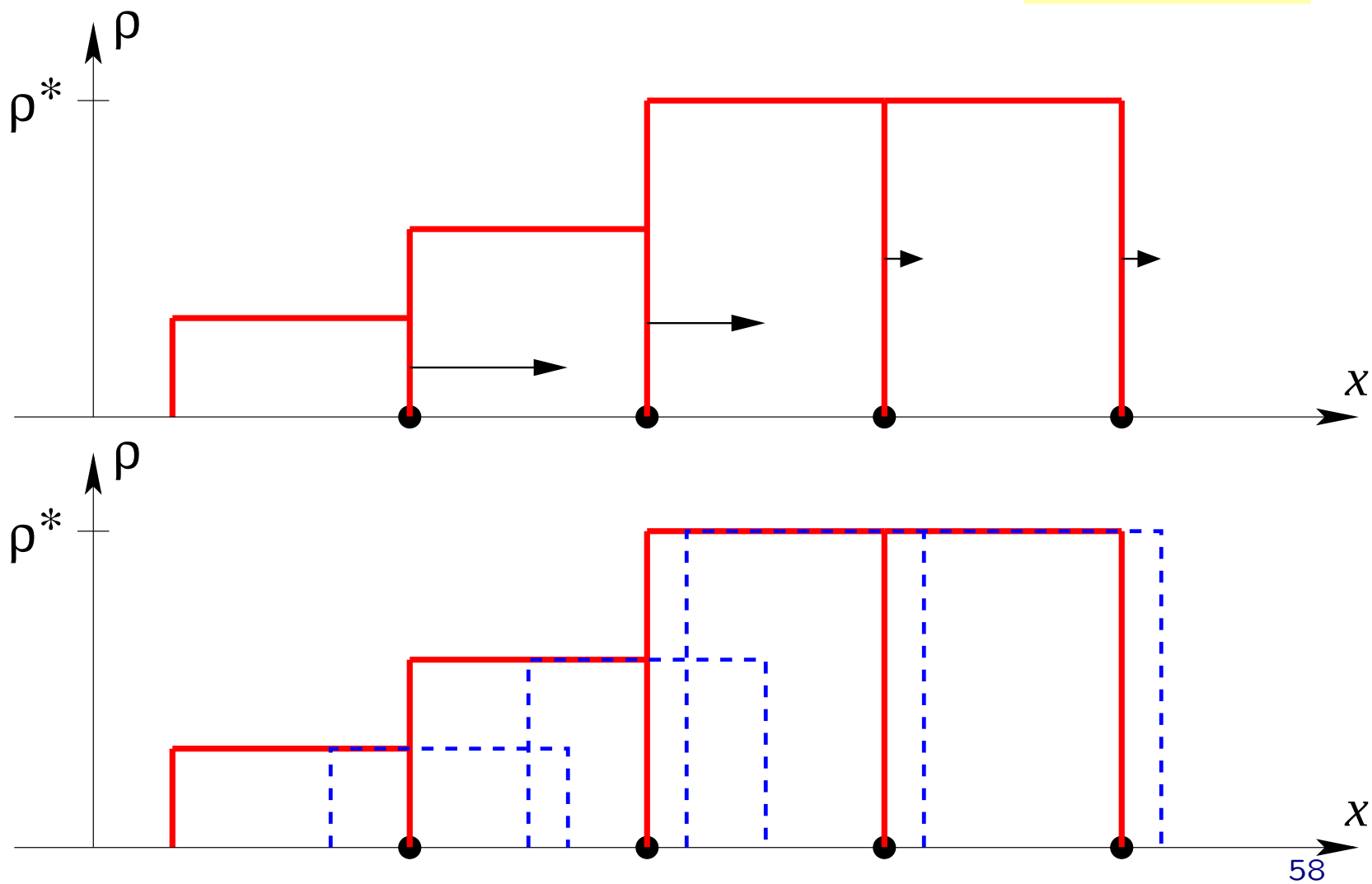
Initial density (left) and velocity (right)



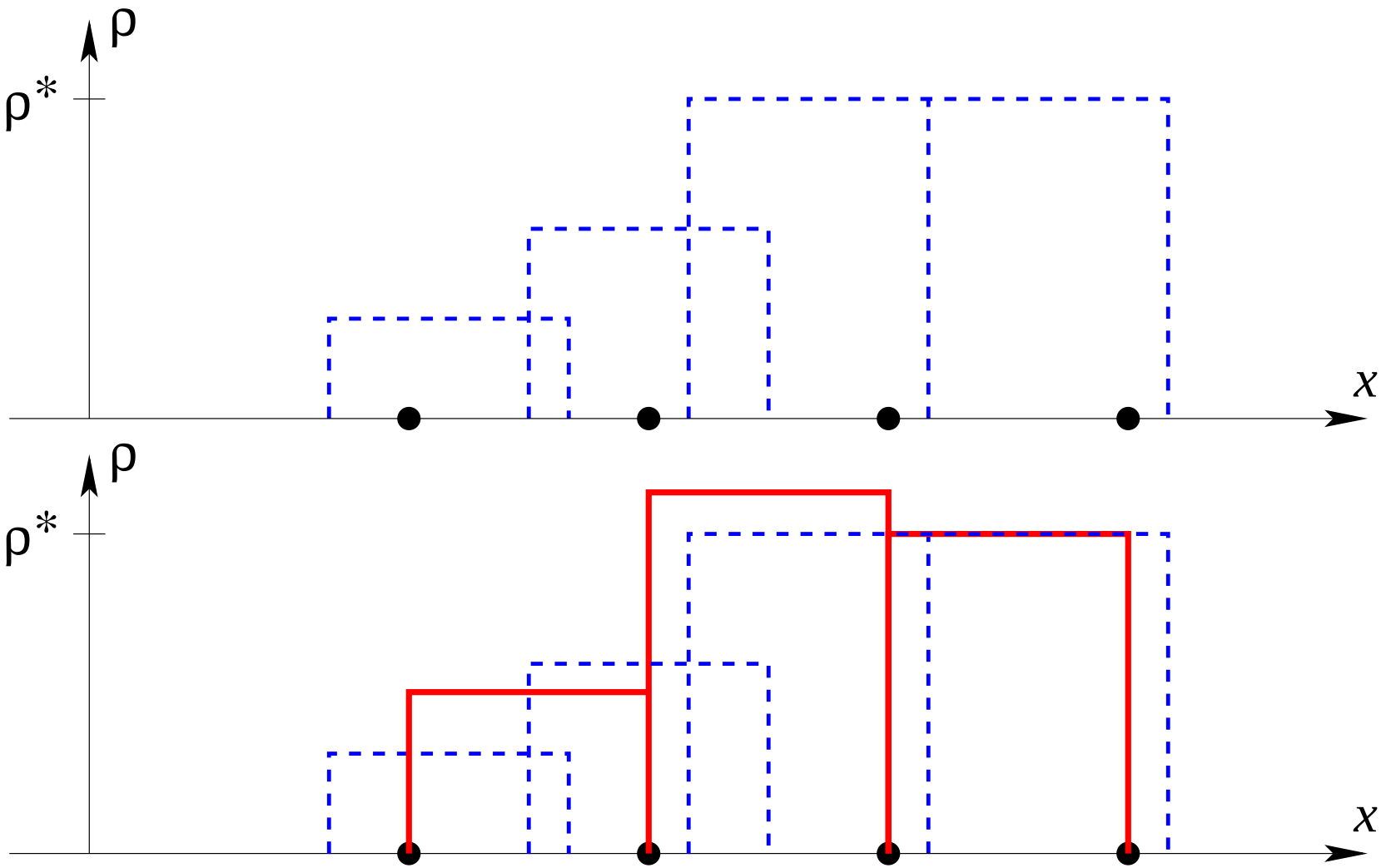
Density at times $t = 25$, 50, 75, and 100

Method 2: Finite-Volume Cluster Propagation

The approximate solution at $t = t^n$ is represented as a piecewise constant function, interpreted as a set of clusters of density ρ_j^n moving with the velocities u_j^n (CFL condition: $\lambda \max_j u_j^n \leq 1$)

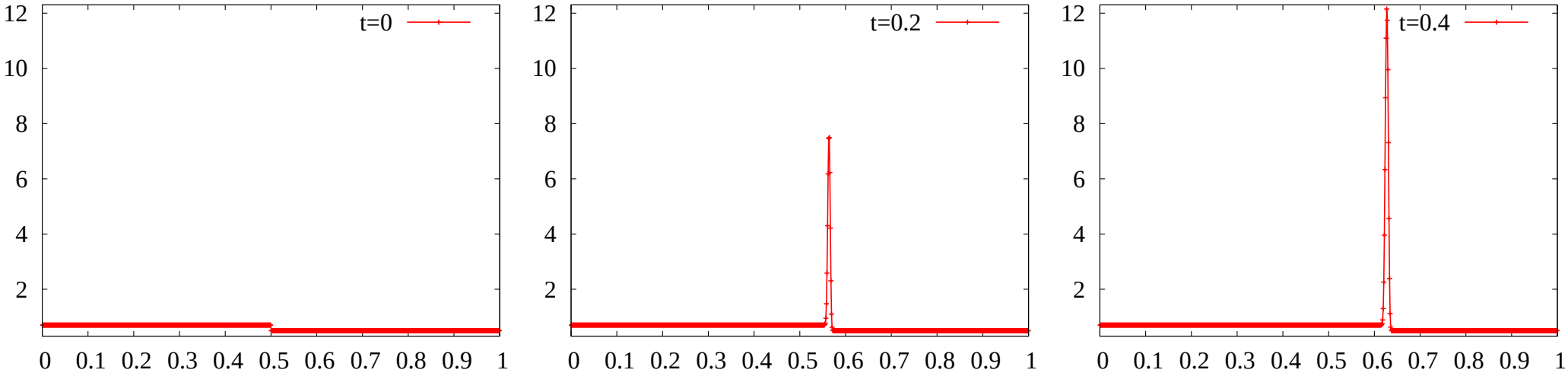


The evolved clusters are then projected onto the original grid

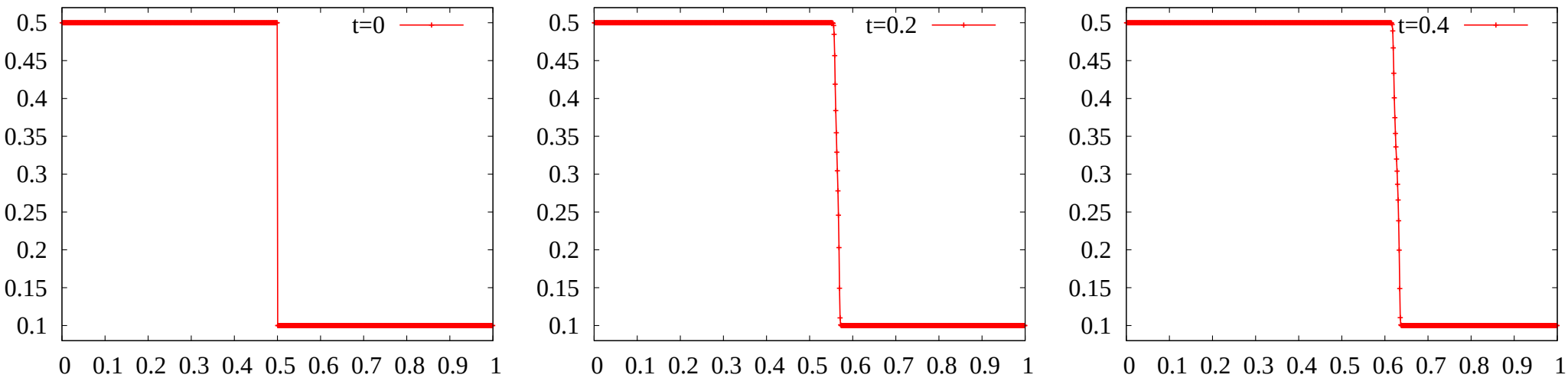


We have obtained the scheme for the PGD.

Example — PGD, Riemann Initial Data (same as in Example 1)

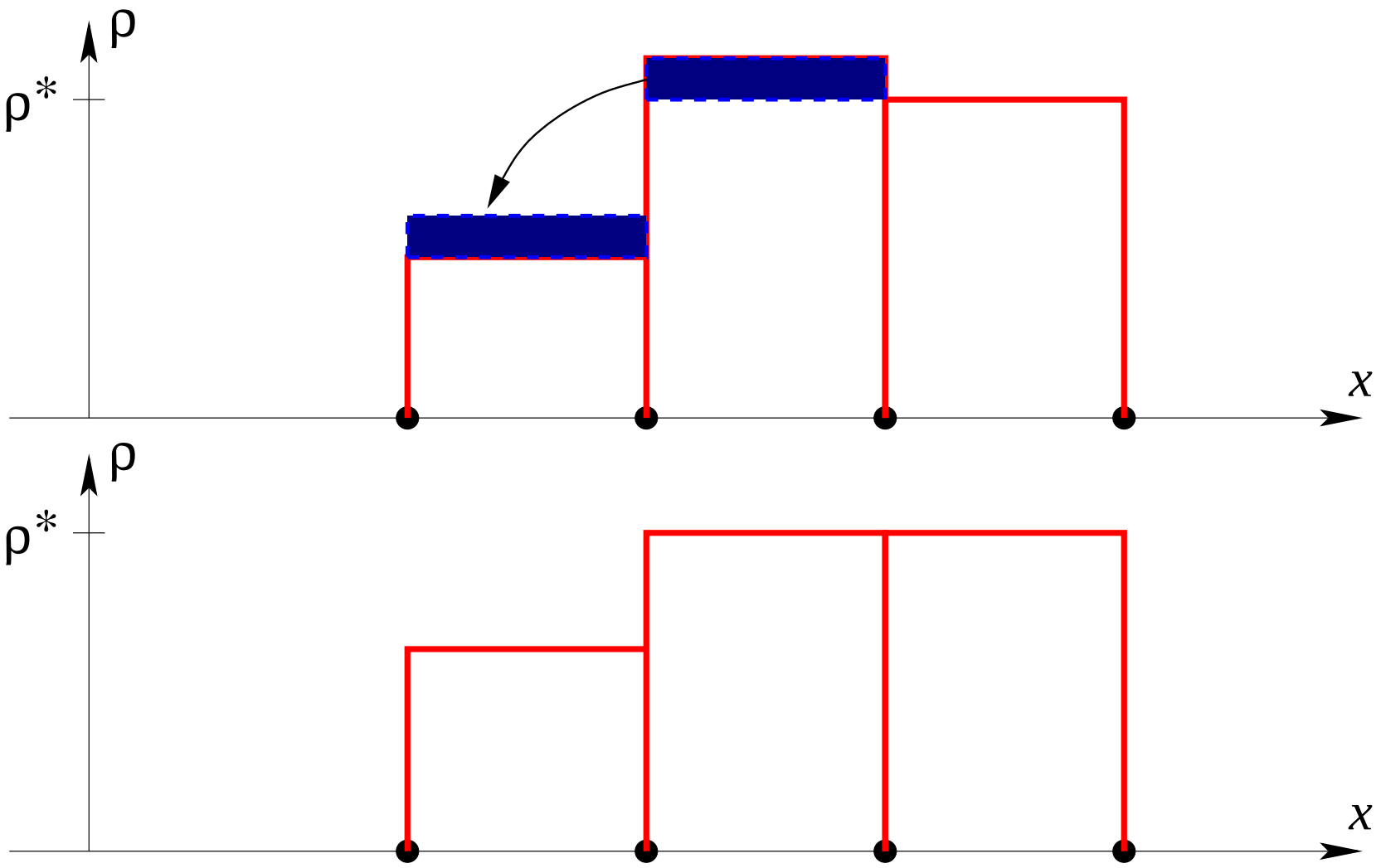


Density

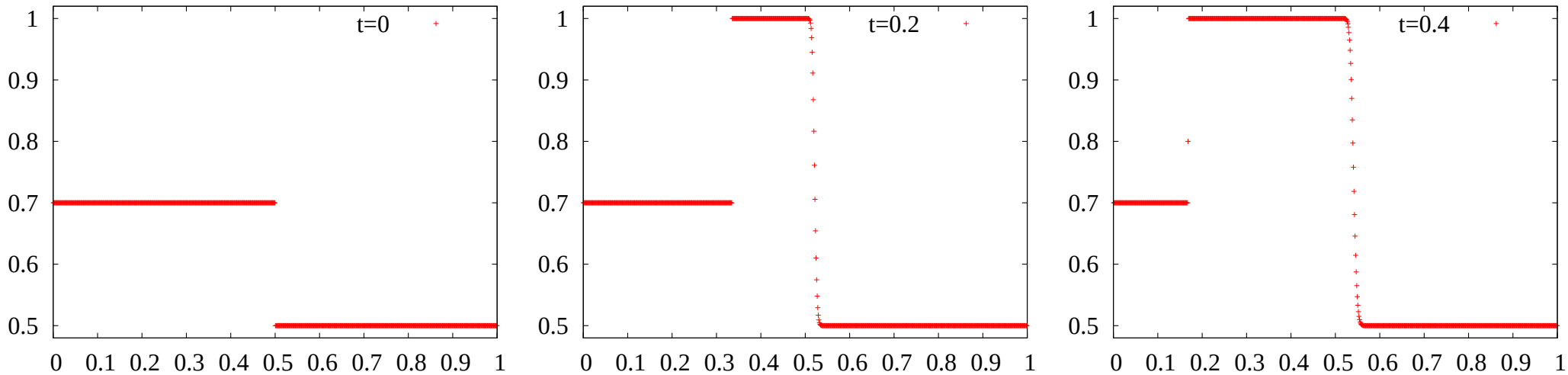


Velocity

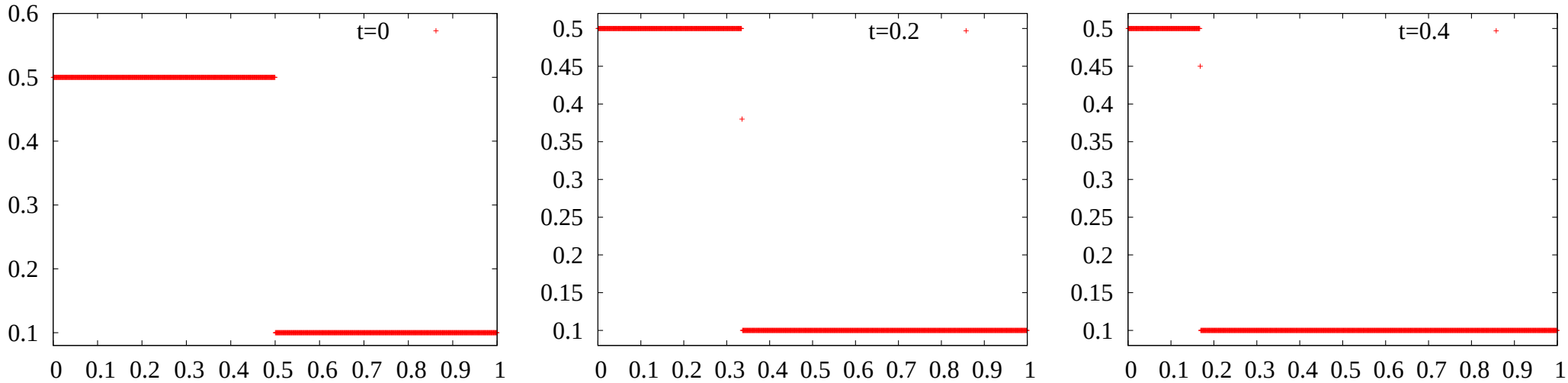
Backward Mass Transfer



Example 1 — Riemann Initial Data

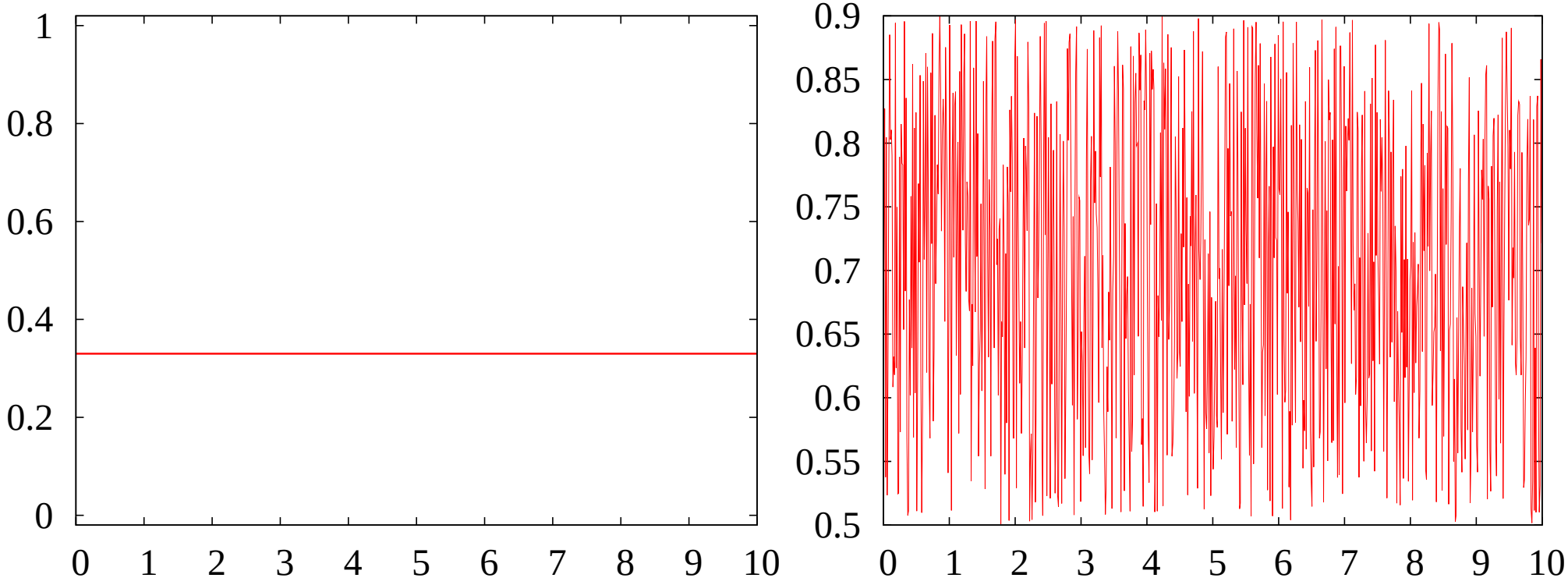


Density

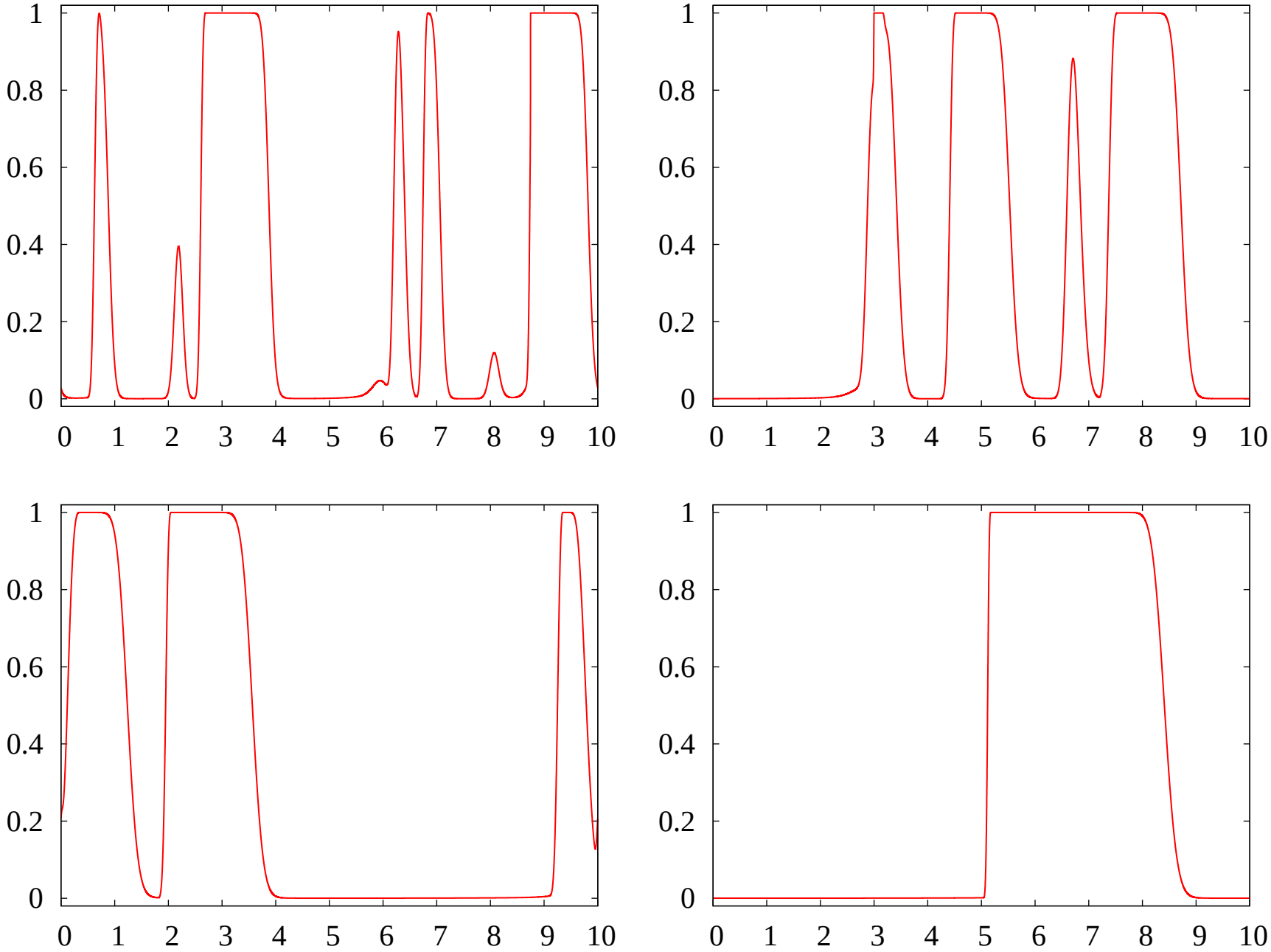


Velocity

Example 2 — Random Initial Velocity



Initial density (left) and velocity (right)



Density at times $t = 70$, 140, 210, and 280

Future Plans

- Second-order extension of the Finite-Volume Cluster Propagation method

- Extension to the case $\rho^* = \rho^*(u)$

- 2-D extension to the models of crowd dynamics