Solution Key for Assignment 7, 412, Applied PDE

12.2.5.b

$$\frac{dx}{dt} = x \Rightarrow x = x_0 e^t \Rightarrow x_0 = xe^{-t}$$
$$\frac{dw}{dt} = 1 \Rightarrow w = t + w_0, \quad w_0 = w(x_0, 0) = f(x_0) = f(xe^{-t})$$
$$w = t + f(xe^{-t})$$

12.2.5.c

$$\frac{dx}{dt} = t \Rightarrow x = \frac{t^2}{2} + x_0 \Rightarrow x_0 = x - \frac{t^2}{2}$$
$$dw = dt \Rightarrow w = t + w_0, \quad w_0 = w(x_0, 0) = f(x_0) = f(x - \frac{t^2}{2})$$
$$w = t + f(x - \frac{t^2}{2})$$

12.2.8

From 12.2.7,

$$u(x,t) = \begin{cases} 1, & x < 2t \\ \frac{x+L}{2t+L}, & 2t < x < 1 + 4t \\ 2, & x > L + 4t \end{cases}$$

take $L \to 0$,

$$u(x,t) = \begin{cases} 1, & x < 2t \\ \frac{x}{2t}, & 2t < x < 1 + 4t \\ 2, & x > L + 4t \end{cases}$$

12.3.5

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$F(x) = -\frac{1}{2c}\int_0^1 g(x)dx = -\frac{1}{2c}\begin{cases} -h, & x < -h \\ x, & -h < x < h \\ h, & x > h \end{cases}$$

$$G(x) = \frac{1}{2c}\int_0^x g(x)dx = -F(x)$$

$$F(x-ct) = \begin{cases} -\frac{h}{2c}, & x < -h+ct \\ \frac{-x+ct}{2c}, & -h+ct < x < h+ct \\ -\frac{h}{2c}, & x > h+ct \end{cases}$$

$$G(x+ct) = \begin{cases} -\frac{h}{2c}, & x < -h-ct \\ \frac{x+ct}{2c}, & -h-ct < x < h-ct \\ \frac{h}{2c}, & x > h-ct \end{cases}$$

$$t > \frac{h}{c} \Rightarrow -h-ct < h-ct < -h+ct < h+ct$$

$$u(x,t) = \begin{cases} 0, & x < -h-ct \\ \frac{x+ct+h}{2c}, & -h-ct < x < h-ct \\ \frac{h}{c}, & h-ct < x < -h+ct \\ \frac{-x+ct+h}{2c}, & -h+ct < x < h+ct \\ 0, & x > h+ct \end{cases}$$

$$t < \frac{h}{c} \Rightarrow -h-ct < -h+ct < h-ct < h+ct$$

$$u(x,t) = \begin{cases} 0, & x < -h-ct \\ \frac{x+ct+h}{2c}, & -h-ct < x < -h+ct \\ t, & -h+ct < x < h-ct \\ \frac{-x+ct+h}{2c}, & h-ct < x < h+ct \\ 0, & x > h+ct \end{cases}$$

12.6.7.a

$$\frac{dx}{dt} = \rho^2 \Rightarrow \rho = \sqrt{\frac{x}{t}}, \quad for \quad 3 < \rho < 4$$

$$\rho(x,t) = \begin{cases} 3, & x < 9t \\ \sqrt{\frac{x}{t}}, & 9t < x < 16t \\ 4, & x > 16t \end{cases}$$

12.6.7.b

$$\frac{dx}{dt} = 4\rho \Rightarrow \rho = \frac{x-1}{4t}, \quad for \quad 2 < \rho < 3$$

$$\rho(x,t) = \begin{cases} 2, & x < 8t+1 \\ \frac{x-1}{4t}, & 8t+1 < x < 12t+1 \\ 3, & x > 12t+1 \end{cases}$$

12.6.7.c

$$\frac{dx}{dt} = 3\rho \Rightarrow \rho = \frac{x}{3t}, \quad for \quad 1 < \rho < 2$$

$$\frac{dx}{dt} = 3\rho \Rightarrow \rho = \frac{x-1}{3t}, \quad for \quad 2 < \rho < 4$$

$$\rho(x,t) = \begin{cases} 1, & x < 3t \\ \frac{x}{3t}, & 3t < x < 6t \\ 2, & 6t < x < 6t+1 \\ \frac{x-1}{3t}, & 6t+1 < x < 12t+1 \\ 4, & x > 12t+1 \end{cases}$$

12.6.7.d

$$\frac{dx}{dt} = 6\rho \Rightarrow \rho = \frac{x}{6t}, \quad for \quad 2 < \rho < 5$$

$$\rho(x,t) = \begin{cases} 2, & x < 3t \\ \frac{x}{6t}, & 12t < x < 30t \\ 5, & x > 30t \end{cases}$$