Math412, Homework 1

Problem 1.3.1. If the rod is hotter than the bath, $u(L,t) - u_B(t) > 0$, then the heat must flow out of the rod at $x = L$. Therefore, the heat flux $\Phi(L,t) = H(u(L,t) - u_B(t)$ is positive which implies that $H > 0$.

Problem 1.4.1. (b) $u(x) = -\frac{T}{L}x + T$; **(g)** $u(x) = -\frac{T}{L+1}x + T$.

Problem 1.4.3.
$u(x) = -\frac{1}{2}x^2 + \frac{2}{3}x$ if $0 \le x \le 1$ and $u(x) = -\frac{1}{6}x + \frac{1}{3}$ if $1 < x \le 2$.

Problem 1.4.10. $E(t) = c\rho\left((4L+1)t + \int_0^L f(x)\,dx\right)$

Problem 1.5.5.
(a) Use Divergence Theorem to show that the heat energy $= 2\pi\int_a^b c\rho u r\,dr$;
(b) Recall that $\Phi = -K_0\nabla u$. Then the flow out is $\Phi \cdot \mathbf{n} = -K_0\frac{\partial u}{\partial r}$ at any point on the boundary with a normal vector $\mathbf{n}$. Integrating over the circle $r = b$, we obtain $-2\pi b K_0 \frac{\partial u}{\partial r}|_{r=b}$.
(c) Apply (1.5.3), using part (a) and (b), to get the result.

Problem 1.5.8. By the Divergence theorem, we get that the integral over the boundary (the closed surface) is equal to

$$\iiint_V \nabla \cdot \nabla u\,dV,$$

where V is the interior of the domain enclosed by the surface. Now, we use Laplace's equation ($\nabla \cdot \nabla u = 0$) and conclude that the integral over the closed surface, the boundary of V (∂V), is equal to zero:

$$\iint_{\partial V} \nabla u \cdot \mathbf{n}\,dS = 0.$$

Problem 1.5.11. For equilibrium, the radial flow at $r = a$, $2\pi a\beta$, must equal the radial flow at $r = b$, $2\pi b$. Thus $\beta = b/a$.
Problem 1.5.14. From calculus we know that the gradient ∇u is perpendicular to isobars (prove it). Insulated boundary means that $\nabla u \cdot \mathbf{n} = 0$ at the boundary, where $\mathbf{n}$ is the unit normal vector to the boundary surface. Therefore, ∇u is perpendicular to both the isobars

and the normal vector $\mathbf{n}$ to the boundary. Hence, the isobars are parallel to the normal of the boundary and therefore perpendicular to the boundary.