Solution Key for Assignment 2, 412, Applied PDE

2.2.2 (a)

$$\begin{aligned} L(c_1u_1+c_2u_2) &= \frac{\partial}{\partial x}[K_0(x)\frac{\partial}{\partial x}(c_1u_1+c_2u_2)] \\ &= \frac{\partial}{\partial x}[c_1K_0(x)\frac{\partial u_1}{\partial x}+c_2K_0(x)\frac{\partial u_2}{\partial x})] \\ &= c_1L(u_1)+c_2L(u_2) \end{aligned}$$

2.2.2(b)

$$\begin{aligned} L(c_1u_1+c_2u_2) &= \frac{\partial}{\partial x}[K_0(x,c_1u_1+c_2u_2)\frac{\partial}{\partial x}(c_1u_1+c_2u_2)] \\ &= \frac{\partial}{\partial x}[c_1K_0(x,c_1u_1+c_2u_2)\frac{\partial u_1}{\partial x}+c_2K_0(x,c_1u_1+c_2u_2)\frac{\partial u_2}{\partial x})] \\ &= c_1\frac{\partial}{\partial x}[K_0(x,c_1u_1+c_2u_2)\frac{\partial u_1}{\partial x}]+c_2\frac{\partial}{\partial x}[K_0(x,c_1u_1+c_2u_2)\frac{\partial u_2}{\partial x}] \\ &\neq c_1L(u_1)+c_2L(u_2) \end{aligned}$$

2.3.2(a)
(i) if $\lambda > 0$, $\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$;
$\phi(0) = 0$ implies $c_1 = 0$, while $\phi(\pi) = 0$ implies $c_2\sin\sqrt{\lambda}x = 0$;
Thus $\sqrt{\lambda}\pi = n\pi$, $n = \lambda^2, n = 1, 2, \ldots$;
(ii) if $\lambda = 0$, $\phi = c_1 + c_2x$;
$\phi(0) = 0$ implies $c_1 = 0$;
$\phi\pi = 0$ implies $c_2 = 0$;
There are no eigenvalues with $\lambda = 0$.
(iii) if $\lambda < 0$, $\phi = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$;
$\phi(0) = 0$ implies $c)_1 = -c_2$;
$\phi\pi = 0$ implies $c_1e^{\sqrt{-\lambda}\pi} + c_2e^{-\sqrt{-\lambda}\pi} = 0$,
which means there are no eigenvalues with $\lambda < 0$.
2.3.2(c)
(i) if $\lambda > 0$, $\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$;
$\frac{d\phi}{dx}(0) = 0$ implies $c_2 = 0$;
$\frac{d\phi}{dx}(L) = 0$ implies $\sqrt{\lambda}L = k\pi$, i.e. $\lambda = \frac{k^2\pi^2}{L^2}$
(ii) if $\lambda = 0$, $\phi = c_1 + c_2x$;
$\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ implies $c_2 = 0$;
$\phi = c_1$.
(iii) if $\lambda < 0$, $\phi = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$;
$\frac{d\phi}{dx}(0) = 0$ implies $c_1 = c_2$;
$\frac{d\phi}{dx}(L) = 0$ implies $\sqrt{\lambda} = 0$
there are no eigenvalues with $\lambda < 0$.
2.3.3(a)
$u(x,t) = \phi(x)G(t)$,
we have $\frac{1}{G}\frac{dG}{dt} = k\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$, it implies $G(t) = c_1e^{-\lambda t}$;
$k\frac{d^2\phi}{dx^2} + \lambda\phi = 0$; if $\lambda > 0$, $\phi(x) = c_1\cos\sqrt{\frac{\lambda}{k}}x + c_2\sin\sqrt{\frac{\lambda}{k}}x$;
$\phi(0) = c_1 = 0$ and $\phi(L) = c_2\sin\sqrt{\frac{\lambda}{k}}L = 0$, thus $\lambda = k(\frac{n\pi}{L})^2, n = 1, 2, \ldots$
Hence $u(x,t) = c_1e^{-\lambda t}c_2\sin\frac{n\pi}{L}x$, from $u(x,0) = 6\sin\frac{9\pi x}{L}$ we get $c_1c_2 = 6$ and $n = 9$;

and $\lambda = k(\frac{9\pi}{L})^2, n = 1, 2, \ldots$
$u(x,t) = 6\sin\frac{9\pi x}{L}e^{-k(\frac{9\pi}{L})^2 t}$
2.3.3(b)
From 2.3.3(a), we have $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$, $\phi(x) = c_2\sin\frac{n\pi x}{L}$, $G(t) = c_1 e^{-\lambda t}$; so that $u(x,t) = c_1c_2\sin\frac{n\pi x}{L}e^{-\lambda t}$, since $u(x,0) = c_1c_2\sin\frac{n\pi x}{L} = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$, we get $u(x,t) = 3\sin\frac{\pi x}{L}e^{-(\frac{\pi}{L})^2 t} - \sin\frac{3\pi x}{L}e^{-(\frac{3\pi}{L})^2 t}$
2.3.5

$$\begin{aligned}
\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx &= \int_0^L \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]dx \\
(n \neq m) \\
&= \frac{1}{2}\frac{L}{(n-m)\pi}\sin(n-m)\pi - \frac{1}{2}\frac{L}{(n+m)\pi}\sin(n+m)\pi \\
&= 0; \\
(n = m) \\
&= \frac{1}{2}L - \frac{1}{2}\frac{L}{(n+m)\pi}\sin(n+m)\pi \\
&= \frac{L}{2}
\end{aligned}$$

2.4.3
(i) if $\lambda > 0$, $\phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$,
by the given conditions, we have $\lambda = k^2$,
hence, $\phi(x) = c_1\cos kx + c_2\sin kx$, $k = 1, 2, \ldots$
(ii) if $\lambda = 0$, it is not an eigenvalue;
(iii) if $\lambda < 0$, it is not an eigenvalue.
2.4.4
Assume $\lambda < 0$, then $\phi(x) = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$;
by the given conditions we have $\sqrt{-\lambda}L = -\sqrt{-\lambda}L$;
Also, $\lambda = 0$ is not the eigenvalue either.
2.4.6
(a) By the given condition, $u(x) = c_1x + c_2$, since $c_1 = 0$, $u(x) = c_2$. as $t \to \infty$, $\int_{-L}^{L} u(x)dx = 2Lc_2$, $c_2 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$.
(b) As $t \to \infty$, $f(x) = a_0$, $2La_0 = \int_{-L}^{L} f(x)dx$, thus $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$.