Verify that: $u_t - \Delta u = \int_{\mathbb{R}^n} f(\bar{x}) \left(G_t - \Delta G \right)(x-\bar{x}, t)\, d\bar{x}$

because G is "smooth" and using $\left| \begin{array}{l} G_t - \Delta G = 0 \\ t > 0 \end{array} \right.$

we conclude $\underline{u_t - \Delta u = 0 \,,\; t > 0}$. V

Now, we solve the non-homogeneous problem

$$\begin{cases} u_t = \Delta u + Q(x,t) \\ u(x,0) = 0 \end{cases}$$

using just like in the one dim. case the solution of

$$\begin{cases} u_t(x,t;\underline{\underline{s}}) = \Delta u(x,t,s) & \underline{t \geq s} \\ u(x,s;\underline{\underline{s}}) = Q(x,s) \end{cases}$$

we build a solution via Duhamel's principle

$$u(x,t) = \int_0^t u(x,t;s)\, ds$$

and because we have a closed form of $u(x,t;\underline{\underline{s}})$ we conclude:

$$u(x,t) = \int_0^t \frac{1}{\left(4\pi(t-s)\right)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x-\bar{x})^2}{4(t-s)}} Q(\bar{x},s)\, d\bar{x}\, ds$$