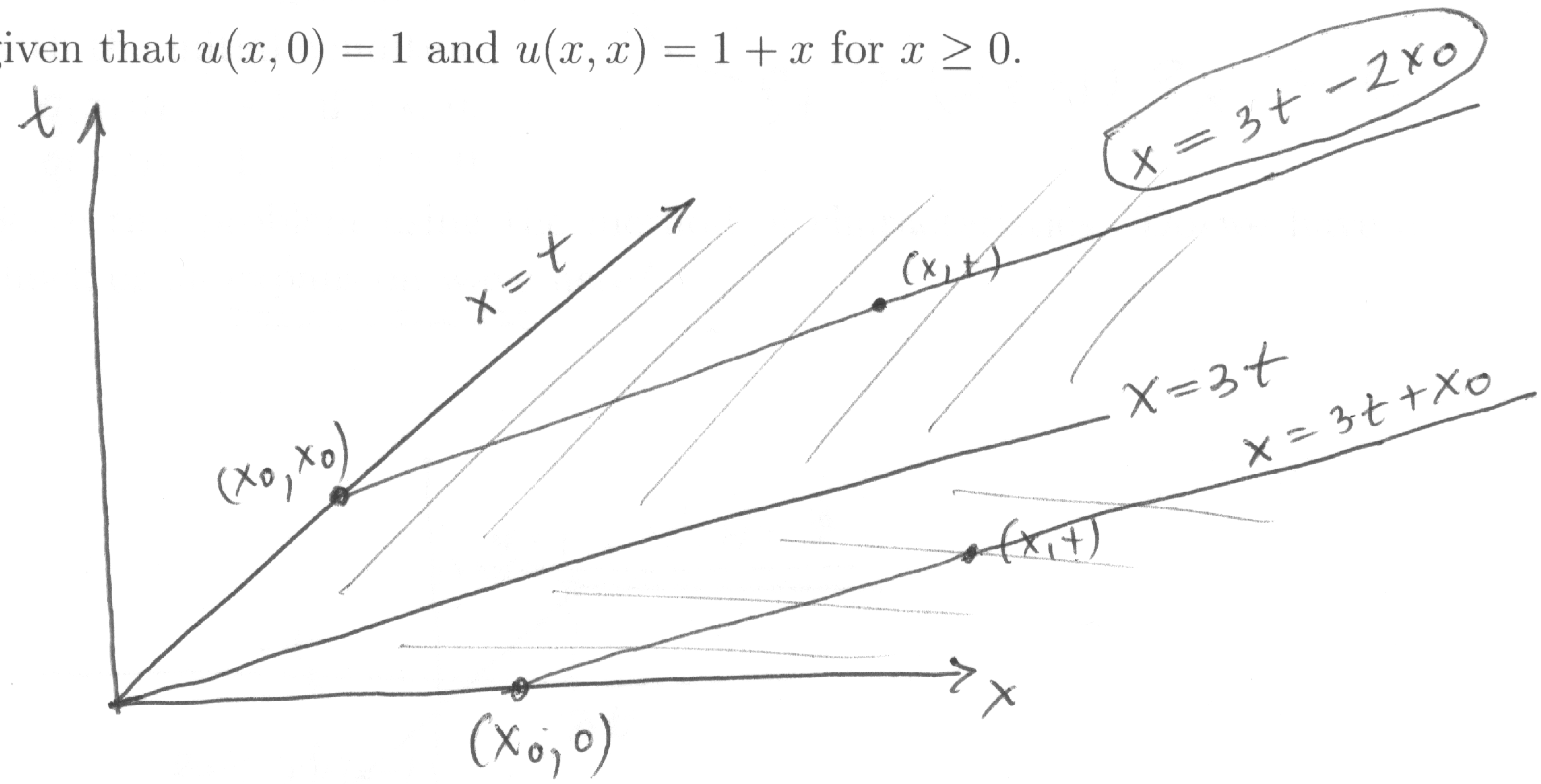
Problem 3. Let $\Omega = \{(x,t) \in \mathbb{R}^2 \,:\, x \geq 0,\ x \geq t\}$. Solve the PDE

$$\partial_t u + 3\partial_x u + 2u = 0 \quad \text{in } \Omega,$$

given that $u(x,0) = 1$ and $u(x,x) = 1 + x$ for $x \geq 0$.



$\dfrac{d}{dt} u(x(t),t) = u_t + u_x \cdot x'(t) = \underline{-2u}$ if $\underline{x'(t) = 3}$

Then $u(x(t),t) = u(x(t_0),t_0)\, e^{-2(t-t_0)}$

because $\dfrac{du}{dt} = -2u$ along $\underline{x(t)}$

① If $\underline{x < 3t}$ then $x - \underline{x_0} = 3(t - \underline{x_0})$ is the characteristic curve $x(t)$, $t \geq x_0$

② If $x > 3t$ then $\underline{x - x_0 = 3t}$ is the curve $x(t)$, $t \geq 0$.

Then $u(x,t) = \begin{cases} \left(1 + \dfrac{3t-x}{2}\right) e^{-2\left(t - \frac{3t-x}{2}\right)} & \text{for } t < x < 3t \\ 1 \cdot e^{-2t} & x > 3t \end{cases}$

Solve for x_0!

i.e. $u(x,t) = \begin{cases} \left(1 + \dfrac{3t-x}{2}\right) e^{t-x} & t < x < 3t \\ e^{-2t} & \text{if } x > 3t \end{cases}$