M602: Methods and Applications of Partial Differential Equations
Mid-Term TEST, Feb 12th
Notes, books, and calculators are not authorized.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1

Let u solve $\partial_t u - \partial_{xx} u = 5$, $x \in (0,L)$, with $\partial_x u(0,t) = \alpha$, $\partial_x u(L,t) = -3$, $u(x,0) = f(x)$.
(a) Let $\alpha = 1$. Compute $\int_0^L u(x,t)dx$ as a function of t.

Integrate the equation over $(0,L)$:

$$d_t \int_0^L u(x,t)dx = \int_0^L \partial_t u(x,t)dx = \int_0^L \partial_{xx} u dx + 5L = \partial_x u|_0^L + 5L.$$

That is $d_t \int_0^L u(x,t)dx = -4 + 5L$. This implies $\int_0^L u(x,t)dx = (-4+5L)t + \int_0^L f(x)dx$.

(b) Let α be an arbitrary number. For which value of α, $\int_0^L u(x,t)dx$ does not depend on t?

The above computation yields $\int_0^L u(x,t)dx = (-3-\alpha+5L)t + \int_0^L f(x)dx$. This is independent of t if $(-3-\alpha+5L) = 0$, meaning $\alpha = -3+5L$.

Question 2

Consider the differential equation $\frac{d^2\phi}{dt^2} + \lambda\phi = 0$, $t \in (0, \pi)$, supplemented with the boundary conditions $\phi(0) = 0$, $\phi'(\pi) = 0$.
(a) What is the sign of λ? Prove your answer.

Multiply the equation by ϕ and integrate over the domain.

$$-\int_0^\pi (\phi'(t))^2 dt + \phi'\phi|_0^\pi + \lambda \int_0^\pi \phi^2(t)dt = 0.$$

Using the BCs, we infer

$$-\int_0^\pi (\phi'(t))^2 dt = \lambda \int_0^\pi \phi^2(t)dt,$$

which means that λ is non-negative.

(b) Compute all the possible eigenvalues λ for this problem and compute ϕ.

There are two cases: Either $\lambda > 0$ or $\lambda = 0$. Assume first $\lambda > 0$, then

$$\phi(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The boundary condition $\phi(0) = 0$ implies imply $c_1 = 0$. The other BC implies $c_2 \cos(\sqrt{\lambda}\pi) = 0$. $c_2 = 0$ gives $\phi = 0$, which is not a proper eigenfunction. The other possibility is $\sqrt{\lambda}\pi = (n + \frac{1}{2})\pi$, $n \in \mathbb{N}$. In conclusion

$$\phi(t) = c_2 \sin((n + \tfrac{1}{2})t), \qquad n \in \mathbb{N}.$$

The case $\lambda = 0$ gives $\phi(t) = c_1 + c_2 t$. The BCs imply $c_1 = c_2 = 0$, i.e. $\phi = 0$, which is not a proper eigenfunction.

Question 3

Consider the Laplace equation $\Delta u = 0$ in the rectangle $x \in [0, L]$, $y \in [0, H]$ with the boundary conditions $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) = 0$, $u(x, H) = f(x)$.
(a) Is there any compatibility condition that f must satisfy for a smooth solution to exist?

f must be such that $f(0) = 0$ and $f(L) = 0$, otherwise u would not be continuous at the two upper corners of the domain.

(b) Solve the Equation.

Use the separation of variable technique. Let $u(x) = \phi(x)\psi(y)$. Then, provided ψ and ϕ are non zero functions, this implies $\frac{\phi''}{\phi} = -\frac{\psi''}{\psi} = \lambda$. Observe that $\phi(0) = \phi(L) = 0$. The usual technique implies that λ is negative. That is to say $\phi(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$. The boundary conditions imply $a = 0$ and $\sqrt{\lambda}L = n\pi$, i.e., $\phi(x) = b\sin(n\pi x/L)$. The fact that λ is negative implies $\psi(y) = c\cosh(\sqrt{\lambda}y) + d\sinh(\sqrt{\lambda}y)$. The boundary condition at $y = 0$ implies $c = 0$. Then the ansatz is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}) \sinh(\frac{n\pi y}{L}),$$

and the usual computation gives

$$A_n = \frac{2}{L\sinh(\frac{n\pi H}{L})} \int_0^L f(\xi) \sin(\frac{n\pi \xi}{L}) d\xi.$$

Question 4

Let $f(x) = x$, $x \in [-L, L]$.
(a) Sketch the graph of the Fourier series of f for $x \in (-\infty, \infty)$.

The Fourier series is equal to the periodic extension of f, except at the points $(2n+1)L$, $n \in \mathbb{Z}$ where it is equal to $0 = \frac{1}{2}(1-1)$.

(b) Compute the Fourier series of f.

f is odd, hence the cosine coefficients are zero. The sign coefficients b_n are given by

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin(\frac{n\pi x}{L})dx = \frac{L}{n\pi}\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})dx - 2\frac{L}{n\pi}\cos(n\pi).$$

As a result $b_n = -2\cos(n\pi)\frac{L}{n\pi} = 2(-1)^{n+1}\frac{L}{n\pi}$ and

$$FS(f)(x) = \sum_{1}^{\infty} 2(-1)^n \frac{L}{n\pi}\sin(\frac{n\pi x}{L}).$$

Question 5

Let L be a positive real number.
(a) Compute the Fourier series of the function $(-L, L) \ni x \longmapsto x^2$.

The Function is even; as a result it sine coefficients are zero. We compute the cosine coefficients as follows:

$$\int_{-L}^{+L} x^2 \cos(m\pi x/L)dx = c_m \int_{-L}^{+L} \cos(m\pi x/L)^2 dx,$$

If $m = 0$, $c_0 = \frac{1}{2L}\int_{-L}^{+L} x^2 dx = \frac{1}{3}L$. Otherwise,

$$c_m = \frac{1}{L}\int_{-L}^{+L} x^2 \cos(m\pi x/L)dx.$$

Integration by parts two times gives

$$\begin{aligned} c_m &= -\frac{1}{L}\frac{L}{m\pi}\int_{-L}^{+L} 2x \sin(m\pi x/L)dx \\ &= 2\frac{L}{m^2\pi^2} x\cos(m\pi x/L)|_{-L}^{L} = 4\frac{L^2}{m^2\pi^2}(-1)^m. \end{aligned}$$

(b) For what values of x the Fourier series is equal to x^2?

The periodic extension of x^2 over $\mathbb{R}$ is piecewise smooth and globally continuous. This means that the Fourier series is equal to x^2 over the entire interval $[-L, +L]$.

(c) Using (a) and (b) give the Fourier series of x over $[-L, +L]$ and say where it is equal to x.

Since we can derivate cosine series, the Fourier series of x over $[-L, +L]$ is obtained by differentiating that of x^2,

$$FS(x)(x) = \frac{d}{dx} FS(\frac{1}{2}x^2)(x) = \sum_{1}^{\infty} 2(-1)^{n+1}\frac{L}{n\pi}\sin(\frac{n\pi x}{L}).$$

We have equality $x = FS(x)(x)$ only on $(-L, +L)$. At $-L$ and $+L$ the Fourier series is zero.

Question 6

Using cylindrical coordinates and the method of separation of variables, solve the Laplace equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{\pi}{2}],\ r \in [0,1]\}$, subject to the boundary conditions $\partial_\theta u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(1,\theta) = \cos(3\theta)$.

We set $u(r,\theta) = \phi(\theta)g(r)$. This means $\phi'' = -\lambda\phi$, with $\phi'(0) = 0$ and $\phi(\frac{\pi}{2}) = 0$, and $rd_r(rd_r g(r)) = \lambda g(r)$. Then using integration by parts plus the boundary conditions we prove that λ is non-negative. Then

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta).$$

The boundary condition $\phi'(0) = 0$ implies $c_2 = 0$. The boundary condition $\phi(\frac{\pi}{2}) = 0$ implies $\sqrt{\lambda}\frac{\pi}{2} = (2n+1)\frac{\pi}{2}$ with $n \in \mathbb{N}$. This means $\sqrt{\lambda} = (2n+1)$. From class we know that $g(r)$ is of the form r^α, $\alpha \geq 0$. The equality $rd_r(rd_r r^\alpha) = \lambda r^\alpha$ gives $\alpha^2 = \lambda$. The condition $\alpha \geq 0$ implies $2n+1 = \alpha$. The boundary condition at $r = 1$ gives $\cos(3\theta) = 1^{2n+1}\cos((2n+1)\theta)$. This implies $n = 1$. The solution to the problem is

$$u(r,\theta) = r^3\cos(3\theta).$$
