Solution Key for Assignment 5, 412, Applied PDE

4.4.9

$$
\begin{aligned}
E &= \frac{1}{2}\int_0^L u_t^2 + c^2 u_x^2 \\
u_{tt} &= c^2 u_{xx} \\
\frac{dE}{dt} &= c^2 \int_0^L (u_t u_{xx} + u_x u_{xt}) dx \\
&= c^2 \int_0^L \frac{d}{dx}(u_x u_t) dx \\
&= c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L
\end{aligned}
$$

4.4.10 (a)

$$
\begin{aligned}
& u(0,t) = u(L,t) = 0 \\
& \frac{dE}{dt} = 0
\end{aligned}
$$

E is a constant.
4.4.10 (b)

$$
\begin{aligned}
& u_x(0,t) = 0, u(L,t) = 0 \Rightarrow u_t(L,t) = 0 \\
\Rightarrow & \frac{dE}{dt} = 0
\end{aligned}
$$

E is a constant.
4.4.10 (c)

$$
\begin{aligned}
& u(0,t) = 0, u_x(L,t) = -\gamma u(L,t), (\gamma > 0) \\
& \frac{dE}{dt} = -\frac{\gamma c^2}{2}\frac{d}{dt}(u(L,t))^2 \\
& E(t) = E(0) + \frac{\gamma c^2}{2}(u(L,0))^2 - \frac{\gamma c^2}{2}(u(L,t))^2
\end{aligned}
$$

E will decrease in time if $u(L,0) = 0$.
4.4.10 (d) If $\gamma < 0$, E will increase in time if $u(L,0) = 0$.
4.4.10 (c) and (d) If we define $E^*(t) := E(t) + \frac{\gamma c^2}{2}(u(L,t))^2$, then this quantity is called the total energy and we need $\gamma > 0$ to make sure that this quantity is nonnegative for arbitrary initial data. Note that $E^*(t)$ is constant in time.
4.4.11

$$
\begin{aligned}
E_p &= \int_0^L \frac{c^2}{2}(\frac{\partial u}{\partial x})^2 dx \\
&= \frac{c^2}{2}\int_0^L [R'(x-ct)]^2 dx; \\
E_k &= \int_0^L \frac{1}{2}(\frac{\partial u}{\partial t})^2 dx \\
&= \frac{1}{2}\int_0^L [-cR'(x-ct)]^2 dx;
\end{aligned}
$$

$$
E_p = E_k
$$

4.4.12

Assume u_1,u_2 are solutions to (4.4.1) to (4.4.3), then we know $u = u_1 - u_2$ is also a solution.

By (4.4.15),

$$\begin{aligned}\frac{dE}{dt} &= c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L \\ E &= c \\ &= \int_0^L \frac{1}{2}(u_t)^2 + \frac{c^2}{2}(u_x)^2 dx\end{aligned}$$

thus,
$E(0) = \int_0^L (\frac{1}{2}0^2 + \frac{c^2}{2}0)dx = 0$

$u_t = 0,\ u_x = 0$
$\Rightarrow u(x,t) = constant$
$u(0,t) = 0,\ u(L,t) = 0,$
$\Rightarrow u(x,t) = 0,$
$\Rightarrow u_1 = u_2.$