ALTERNATIVE APPROACH ON COVERING PROBABILITY
WHEN D=3
(TECHNICAL REPORT)

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ABSTRACT. In this technical report we proved the alternative approach to show an upper bound for the probability that a 3 dimensional simple random walk covers each point in a nearest neighbor path connecting 0 and the boundary of a $L_1$ ball of radius $N$ described in [4]. Although presently this approach gives a weaker upper bound that $P(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty)) \leq C \exp(-cN^{1/3})$, it might have the potential to obtain a sharp exponential asymptotic.

1. INTRODUCTION

In this technical report, we give a detailed proof of Proposition 4.1 in [4] which gives an alternative approach to show an upper bound for the covering probability of a 3 dimensional simple random walk. In this report, for any integer $N \geq 1$, let $\partial B_1(0, N)$ be the boundary of the $L_1$ ball in $\mathbb{Z}^3$ with radius $N$. We call a nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K)$ connecting 0 and $\partial B_1(0, N)$ if $P_0 = 0$ and $\inf\{n : \|P_n\|_1 = N\} = K$. And we call that a path $\mathcal{P}$ is covered by a $d$ dimensional random walk $\{X_{d,n}\}_{n=0}^\infty$ if

$$\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(X_{d,0}, X_{d,1}, \cdots).$$

Here we prove

**Theorem 1.1.** (Proposition 4.1 in [4]) There are $c, C \in (0, \infty)$ such that for any nearest neighbor path $\mathcal{P} = (P_0, P_1, \cdots, P_K) \subset \mathbb{Z}^3$ connecting 0 and $\partial B_1(0, N)$

$$P(\text{Trace}(\mathcal{P}) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty)) \leq C \exp\left(-cN^{1/3}\right).$$

Although this upper bound is much weaker than Theorem 1.3 in [4] (which is proved using a completely different approach), the reason we want to present this proof is that when looking at the locations at each time $X_{3,n}$ returns to the diagonal, we might be able to use the same technique here to obtain a sharp upper bound of $\exp(-cN)$ for $d = 3$. See Conjecture 4.1 in [4] for details.

The structure of this report is as follows: In Section 2, we show the tail asymptotic of the returning probability of $\{\hat{X}_{2,n}\}_{n=0}^\infty$. In Section 3, we find the proper time $T$ so that $\{X_{3,n}\}_{n=T}^\infty \cap B_2(0, N) = \emptyset$ with probability $1 - O(\exp(-cN^{-1/3})).$
With these results, in Section 4, we can properly truncate the excursions of $\hat{X}_{2,n}$ and show Theorem 1.1 using large deviation techniques.

2. Tail asymptotic on the excursion of $\{\hat{X}_{2,n}\}_{n=0}^\infty$

In this section, we find the tail asymptotic on the excursion of $\{\hat{X}_{2,n}\}_{n=0}^\infty$. For 2 dimensional simple random walk $\{X_{2,n}\}_{n=0}^\infty$ and stopping time

$$\tau_{2,1} = \inf\{n \geq 1 : X_{2,n} = 0\},$$

it was shown in [1] and [2] that

$$(2.1) \quad P(\tau_{2,1} > n) = \frac{\pi}{\log(n)} + O\left(\frac{1}{\log^2(n)}\right)$$

as $n \to \infty$. Here we will use local central limit theorem together with an argument from [2] to show that the same order of decay holds for our non simple random walk. Define the stopping times

$$\tau_{l,3,0} = 0 \quad \tau_{l,3,1} = \inf\{n \geq 1 : \hat{X}_{2,n} = 0\}$$

and

$$\tau_{l,3,i} = \inf\{n > \tau_{l,3,i-1} : \hat{X}_{2,n} = 0\}$$

for all $i \geq 2$.

**Lemma 2.1.** For the constant

$$C_1 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp\left[-\frac{1}{3}(s_1^2 - s_1 s_2 + s_2^2)\right] ds_1 ds_2,$$

$$(2.2) \quad P(\tau_{l,3,1} > n) = \frac{1}{C_1 \log(n)} + O\left(\frac{1}{\log^2(n)}\right)$$

as $n \to \infty$.

**Proof.** For any $n$ using the same argument on page 139 of [2], we can apply the total probability theorem on $\sup\{t \leq n : \hat{X}_{2,t} = 0\}$ and have

$$(2.3) \quad \sum_{m=0}^{n} p_{2,m}(0) P(\tau_{l,3,1} > n - m) = 1$$

where $p_{2,m}(0) = P(\hat{X}_{2,m} = 0)$. For $p_{2,0}(0)$, note that $\{\hat{X}_{2,n}\}_{n=0}^\infty$ is a finite range random walk with covariance matrix

$$\Gamma = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Then since

$$p_{2,2}(0) \geq P(X_{3,1} = e_{3,1}, X_{3,2} = -e_{3,1}) = \frac{1}{36} > 0$$
and

\[ p_{2,3}(0) \geq P(X_{3,1} = e_{3,2}, X_{3,2} = e_{3,1}, X_{3,3} = e_{3,3}) = \frac{1}{216} > 0, \]

one can immediately see that \( \{\hat{X}_{2,n}\}_{n=0}^\infty \) is aperiodic. Thus by local central limit theorem (see Equation (2.5) and Theorem 2.1.1 in [3] for details), let

\[ \bar{p}_{2,m}(0) = \frac{1}{(2\pi)^2 m} \int_{\mathbb{R}^2} \exp \left[ -\frac{1}{3}(s_1^2 - s_1 s_2 + s_2^2) \right] ds_1 ds_2. \]

There is a \( c < \infty \) such that for all integers \( m > 0 \),

(2.4) \[ |\bar{p}_{2,m}(0) - p_{2,m}(0)| \leq \frac{c}{m^2}. \]

Thus we have

(2.5) \[ p_{2,m}(0) \in \left[ \frac{C_1}{m} - \frac{c}{m^2}, \frac{C_1}{m} + \frac{c}{m^2} \right] \]

where

\[ C_1 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp \left[ -\frac{1}{3}(s_1^2 - s_1 s_2 + s_2^2) \right] ds_1 ds_2 \]

is a constant. Combining (2.3), (2.4) and (2.5), and noting that \( \{P(\tau_{l,1} > n)\}_{n=0}^\infty \) is a monotonically non increasing sequence, we have that

\[ P(\tau_{l,1} > n) \left( \sum_{m=0}^{n} p_{2,m}(0) \right) \leq 1 \]

and that there is a \( C_2 < \infty \) such that for any \( 0 \leq n' < n \)

(2.6) \[ \sum_{m=0}^{n} p_{2,m}(0) \in [C_1 \log(n) - C_2, C_1 \log(n) + C_2], \]

and

(2.7) \[ \sum_{m=n'+1}^{n} p_{2,m}(0) \in [C_1 \log(n/n') - C_2, C_1 \log(n/n') + C_2], \]

which implies that for sufficiently larger \( n \)

(2.8) \[ P(\tau_{l,1} > n) \leq \frac{1}{C_1 \log(n) - C_2} \leq \frac{1}{C_1 \log(n)} + \frac{2C_2}{C_1^2 \log^2(n)}. \]

Then to find an upper bound, note that for any \( 1 < m_1 < m_2 < 2n \), we always have

(2.9) \[ P(\tau_{l,1} > 2n-m_1) \left( \sum_{m=0}^{m_1} p_{2,m}(0) \right) + P(\tau_{l,1} > 2n-m_2) \left( \sum_{m=m_1+1}^{m_2} p_{2,m}(0) \right) + \sum_{m=m_2+1}^{2n} p_{2,m}(0) \geq 1. \]
Now let $m_1 = n$, $m_2 = \lceil 2n - n/\log(n) \rceil$, we can apply (2.7) and (2.8) to the second term in (2.9) and have
\[
\sum_{m=m_1+1}^{m_2} p_{2,m}(0) \leq C_1 \log(2) + C_2
\]
while for sufficiently large $n$
\[
P(\tau_{l,1} > 2n - m_1) \leq P(\tau_{l,1} \geq n/\log(n)) \leq \frac{2}{C_1 \log(n)}.
\]
Thus there is a $C_3 < \infty$ such that
\[
(2.10) \quad P(\tau_{l,1} > 2n - m_2) \left( \sum_{m=m_1+1}^{m_2} p_{2,m}(0) \right) \leq C_3 \log(n).
\]
Then for the third term in (2.9), noting that by (2.5) we have
\[
(2.11) \quad \sum_{m=m_2+1}^{2n} p_{2,m}(0) \leq \sum_{m=m_2+1}^{2n} \frac{C_1}{m} + \frac{c}{m^2} \leq \frac{C_1}{\log(n)},
\]
for sufficiently large $n$. Thus combining (2.6), (2.9), (2.10) and (2.12), we have
\[
(2.12) \quad P(\tau_{l,1} > n) \geq \frac{1 - \frac{C_1 + C_3}{\log(n)}}{C_1 \log(n) + C_2} = \frac{1}{C_1 \log(n)} + O\left(\frac{1}{\log^2(n)}\right),
\]
and the proof of this lemma is complete. \(\square\)

3. Last exiting time of 3 dimensional simple random walk

In this section, we study the last exiting time of 3 dimensional simple random walk $\{X_{3,n}\}_{n=0}^{\infty}$ and show that

**Lemma 3.1.** For $T^0_N = \lceil \exp(N^{1/3}) \rceil$, we have
\[
P(\{X_{3,n}\}_{n=T^0_N}^{\infty} \cap B_2(0,N) \neq \emptyset) \leq \exp(-N^{1/3}/5)
\]
for all $N$ larger enough.

**Proof.** Let $R_N = \exp(N^{1/3}/4)$. We can define the stopping times
\[
T_N = \inf\{n : \|X_{3,n}\|_2 > R_N\}
\]
and
\[
T_{2,N} = \inf\{n : \|X_{3,n}\|_2 > 2R_N\}
\]
and have
\[
(3.1) \quad P(\{X_{3,n}\}_{n=T^0_N}^{\infty} \cap B_2(0,N) \neq \emptyset) \leq P(T_N \leq T^0_N) P(\{X_{3,n}\}_{n=T_N}^{\infty} \cap B_2(0,N) \neq \emptyset | T_N \leq T^0_N) + P(T_N > T^0_N).
\]
It is well known that there is a $c_3 < \infty$ such that for any $x \in \mathbb{Z}^3$

\[(3.2)\]

\[P\left(x \in \{X_{3,n}\}_{n=T_N}^{\infty}\right) \leq \frac{c_3}{\|x\|_2}.\]

Thus there is a $C_3 < \infty$ such that for any sufficiently large $N$,

\[(3.3)\]

\[P(T_N \leq T^0_N) \leq C_3 N^3 R_N \leq \frac{1}{2} \exp(-N^{1/3}/5).\]

To control $P(T_N \geq T^0_N)$, let $t_{N,i} = i \exp(N^{1/3}/2)$, $i = 0, 2, \cdots, \exp(N^{1/3}/2)$. Note that, by central limit theorem, there is a $c_0 > 0$ such that for any $i$ and any sufficiently large $N$

\[(3.4)\]

\[P(T_N > t_{N,i+1}|T_N > t_{N,i}) \leq P(T_{2N} \geq \exp(N^{1/3}/2)) \leq 1 - c_0.\]

Thus

\[(3.5)\]

\[P(T_N \geq T^0_N) = \prod_{i=0}^{\exp(N^{1/3}/2)-1} P(T_N > t_{N,i+1}|T_N > t_{N,i}) \leq (1 - c_0)^{\exp(N^{1/3}/2)} \leq \frac{1}{2} \exp(-N^{1/3}/5).\]

Combining (3.1), (3.3) and (3.5), the proof of Lemma 3.1 is complete.

\[\square\]

4. PROOF OF THEOREM 1.1

Recalling that $T^0_N = \lceil \exp(N^{1/3}) \rceil$, denote the constants $C_N = T^0_N / N$ and $M_N = 8C_1 C_{N} \log(C_{N})$. Then recalling that $\tau_{l_3,0} = 0$

\[\tau_{l_3,1} = \inf\{n \geq 1 : \hat{X}_{2,n} = 0\}\]

and

\[\tau_{l_3,i} = \inf\{n > \tau_{l_3,i-1} : \hat{X}_{2,n} = 0\}\]

for all $i \geq 2$. Define truncated excursion random variables

\[Y_i^N = \min\{\tau_{l_3,1} - \tau_{l_3,i-1}, M_N\}, \; i = 1, 2, \cdots.\]

Then by definition one can immediately see that $Y_i^N$, $i = 1, 2, \cdots$ is an i.i.d. sequence of random variables and that

\[\tau_{l_3,n} \geq \sum_{i=1}^{n} Y_i^N.\]
Moreover, by Lemma 2.1
\[ E[Y_i] = \sum_{n=1}^{M_N} P(\tau_{i,1} \geq n) \geq M_N P(\tau_{i,1} \geq M_N) \]
(4.1)
\[ \geq \frac{7.5C_1C_N \log(C_N)}{C_1 [\log(3C_1) + \log(C_N) + \log(\log(C_N))]}) \geq 7C_N. \]

And we have
(4.2) \[ \text{var}(Y_i) \leq M_N^2 \]
as a result of the uniform upper bound \(|Y_i| \leq M_N|.

At this point, we have all the intermediate results we need to prove Theorem 1.1. Consider the events
(4.3) \[ A_1 = \{ \{X_{3,n}\}_{n=T_0^N}^\infty \cap B_2(0, N) \neq \emptyset \} \]
and
(4.4) \[ A_2 = \{ \tau_{i,|[N/3]} < T_0^N \}. \]

It is easy to see that
(4.5) \[ P( \text{Trace}(P) \subseteq \text{Trace}(\{X_{3,n}\}_{n=0}^\infty)) \leq P(A_1 \cup A_2) \leq P(A_1) + P(A_2). \]

In Lemma 3.1 we have already shown that \( P(A_1) \leq \exp(-N^{1/3}/5). \) So the rest of the proof will concentrate on controlling \( P(A_2). \) Define
\[ A_2^N = \left\{ \sum_{i=1}^{[N/3]} Y_i^N < T_0^N \right\} \]
and noting that
\[ \sum_{i=1}^{[N/3]} Y_i^N \leq \tau_{i,|[N/3]}, \]
one can immediately see that \( P(A_2) \leq P(A_2^N). \) For each \( i \) let \( \bar{Y}_i^N = Y_i^N - E[Y_i^N], \) then recalling that \( C_N = T_0^N/N \) and that \( E[Y_i^N] \geq 7C_N, \) we have
\[ A_2^N = \left\{ \sum_{i=1}^{[N/3]} Y_i^N < T_0^N \right\} = \left\{ \sum_{i=1}^{[N/3]} \bar{Y}_i^N < T_0^N - [N/3]E[Y_i^N] \right\} \subset \left\{ \sum_{i=1}^{[N/3]} \bar{Y}_i^N < -T_0^N \right\}. \]

Noting that now all \( \bar{Y}_i^N \)'s are unbiased, by Azuma-Hoeffding inequality,
(4.6) \[ P \left( \sum_{i=1}^{[N/3]} \bar{Y}_i^N < -T_0^N \right) \leq \exp \left( -\frac{(T_0^N)^2}{2[N/3]M_N^2} \right). \]
Recalling that $T_{N}^{0} = \exp\left(\frac{N}{3}\right) = NC_N$ and that $M_N = 8C_1C_N\log(C_N)$, there is a $c > 0$ such that for sufficiently large $N$

$$P(A_2) \leq \exp\left(-\frac{(T_{N}^{0})^2}{[N/3]M_N^2}\right) \leq \exp\left(-c\frac{N}{\log^2(C_N)}\right)$$

$$\leq \exp\left(-c\frac{N}{\log^2(T_{N}^{0})}\right) = \exp(-cN^{1/3}).$$

Combining (4.7) and Lemma 3.1, the proof of Theorem 1.1 is complete. □

**References**


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