A transcendence criterion for CM on some families of Calabi-Yau manifolds

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Abstract

In this paper, we give some examples of the validity of a special case of a recent conjecture of Green, Griffiths and Kerr [7]. This special case is an analogue of a celebrated theorem of Th. Schneider [17] on the transcendence of values of the elliptic modular function, and its generalization in [4], [19]. Related techniques apply to all the examples of CMCY families in the work of Rohde [15], and this is the subject of a paper in preparation by the author [22].[1]

1 Introduction

In a recent monograph [7], Green, Griffiths and Kerr propose a general theory of Mumford–Tate domains in which to examine new problems on arithmetic, geometry and representation theory, generalizing the well-established results of the theory of Shimura varieties. In the last section of this monograph, they formulate an algebraic independence conjecture for points in period domains originating in one of Grothendieck, and based on ideas of André for Shimura varieties. (André's ideas are subsummed in Grothendieck's Conjecture, as has been thoroughly explained in the unpublished thesis of A. Präve [14]. Nonetheless, André's formulation is very useful for the context of [7]).

Very little is proven about algebraic independence of periods on abelian varieties defined over number fields. However, much has been established about the special case of their linear independence properties. Using these

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properties, we can deduce results about the transcendence of automorphic functions at algebraic points.

The first important result of this type is due to Th. Schneider [17] in 1937. Let $\mathcal{H}$ be the upper half plane, namely the complex numbers with positive imaginary part. Let $j(\tau)$, $\tau \in \mathcal{H}$, be the elliptic modular function, which is the unique function, automorphic with respect to $\mathrm{PSL}(2,\mathbb{Z})$, holomorphic with a simple pole at infinity, and with Fourier series of the form

$$j(\tau) = e^{-2\pi i\tau} + 744 + \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}, \qquad a_n \in \mathbb{C}.$$

Th. Schneider proved that

$$\{\tau \in \mathcal{H} \cap \overline{\mathbb{Q}} : j(\tau) \in \overline{\mathbb{Q}}\} = \{\tau \in \mathcal{H} : [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}.$$

Therefore $j(\tau)$ is a transcendental number for all $\tau \in \mathcal{H} \cap \overline{\mathbb{Q}}$ which are not imaginary quadratic, that is, are not complex multiplication (CM) points. We view this as a transcendence criterion for complex multiplication, not only because it is equivalent to a statement about transcendence of special values of automorphic functions, but, more importantly, because the proof uses techniques from transcendental number theory. The analogous result for Shimura varieties of PEL type is due to the author, jointly with Shiga and Wolfart [4], [19]. There, the key transcendence technique is the Analytic Subgroup Theorem of Wüstholz [27]. Recall that to every polarized abelian variety A of complex dimension g, we can associate a normalized period matrix τ_A in the Siegel upper half space $\mathcal{H}_g$ of genus g, consisting of the $g \times g$ symmetric matrices with positive definite imaginary part. Then, the results of [4], [17], [19] are equivalent to the statement that A is defined over $\overline{\mathbb{Q}}$ as an algebraic variety and τ_A has matrix coefficients algebraic numbers if and only if A has complex multiplication (CM). Of course, the matrix τ_A is only defined up to the action on $\mathcal{H}_g$ of the integer points of a symplectic group, which does not affect the statement.

The simplest case of Conjecture (VIII.A.8) of [7] asks for similar results for variations of Hodge structure of level $n \geq 1$ (the PEL (or Hodge type) Shimura variety case being level 1). In this paper, we prove such results in certain examples, namely for families of Calabi-Yau threefolds shown by Borcea [2] and Viehweg–Zuo [23] to have Zariski dense sets of complex multiplication fibers. We also indicate how to treat the first step of a tower construction of Calabi–Yau manifolds due to Borcea [3] and Voisin [24]. Similar considerations in [22], where full details will be given, enable us to treat all the examples of Rohde in [15].

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2 The problem and the main results

In this section, we describe the problem we are studying. We then mention briefly the families of Calabi-Yau manifolds, proved by Rohde [15] to have dense sets of CM fibers, for which the problem can be solved [22]. After that, we focus for the rest of the paper on the examples of Borcea [2], of Viehweg–Zuo [23], and the first step of what Rohde calls a "Borcea–Voisin" tower [3], [24].

A Calabi–Yau n-fold X is defined as a complex compact Kähler manifold with $H^{k,0}(X) = \{0\}$, $k = 1, \ldots, n-1$, and a nowhere vanishing holomorphic n-form.

For the convenience of the reader, we first recall some basic definitions from Hodge theory. They are well-documented in literature spanning many years, and can be found in [7]. For a $\mathbb{Q}$-vector space V and a field $k \supseteq \mathbb{Q}$, we denote $V_k = V \otimes_{\mathbb{Q}} k$ and $\mathrm{GL}(V)_k = \mathrm{GL}(V_k)$. A Hodge structure of level $n \in \mathbb{Z}$ is a finite dimensional $\mathbb{Q}$-vector space V, endowed with the following three equivalent things:

- a decomposition of vector spaces $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$, with $V^{p,q} = \overline{V}^{q,p}$.
- a filtration $F^n \subset F^{n-1} \subset \ldots \subset F^0 = V_{\mathbb{C}}$, with $F^p \oplus \overline{F}^{n-p+1} \simeq V_{\mathbb{C}}$.
- a homomorphism of $\mathbb{R}$-algebraic groups

  $$\varphi : \mathbb{U}(\mathbb{R}) \to \mathrm{SL}(V)_{\mathbb{R}}$$

  with $\varphi(-\mathrm{Id}_{\mathbb{U}}) = (-1)^n \mathrm{Id}_V$. Here $\mathbb{U}$ is the group, whose k-points, where $k \supseteq \mathbb{Q}$ is a field, are

  $$\mathbb{U}(k) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1,\, a, b \in k \right\}.$$

For $z \in \mathbb{C}$ with $|z| = 1$, we have $\varphi(z) = z^{p-q}$ on $V^{p,q}$, where $z = a + ib$, $a, b \in \mathbb{R}$, is identified with the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{U}(\mathbb{R})$. The endomorphism $C = \varphi(i)$ is called the Weil operator. The $\mathbb{Q}$-vector space $V = \mathbb{Q}$ is assumed to have the trivial Hodge structure φ_{triv} of weight 0 which maps $\mathbb{U}(\mathbb{R})$ to Id_V. A Hodge structure (V, φ) is polarized if there is a bilinear non-degenerate map

$$Q : V \otimes V \to \mathbb{Q},$$

with

$$Q(u, v) = (-1)^n Q(v, u) \tag{1}$$

satisfying the Hodge-Riemann (HR) relations

$$Q(F^p, F^{n-p+1}) = 0, \qquad \text{(HR1)},$$

$$Q(u, C\,\overline{u}) > 0, \quad u \neq 0, \quad u \in V_{\mathbb{C}}, \qquad \text{(HR2)}.$$

Let $G = \text{Aut}(V, Q)$, and denote by $G(k)$, $\mathbb{Q} \subseteq k$ a field, the k-points of G. Usually there will be a lattice $V_{\mathbb{Z}}$ with $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$, so that $G(\mathbb{Z})$ is the arithmetic subgroup of G preserving $V_{\mathbb{Z}}$.

The Mumford-Tate group (MT) M_φ of a Hodge structure (V, φ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\text{SL}(V)$ whose real points contain $\varphi(\mathbb{U}(\mathbb{R}))$. Here, we have used the terminology of [7], rather than calling this the Hodge group or special Mumford–Tate group. A Hodge structure (V, φ) is called a CM (complex multiplication) Hodge structure if and only if its Mumford-Tate group is abelian. We just say φ is CM, if the intended V is clear from the context, or just say V is CM, if the intended φ is clear from the context. We refer to $(\mathbb{Q}, \varphi_{\text{triv}})$ as the trivial CM Hodge structure.

Let a $\mathbb{Q}$-vector space V and a non-degenerate bilinear form Q satisfying (1) be given. Furthermore, for all integers p, q with $p + q = n$, let integers $h^{p,q} \geq 0$ with $h^{p,q} = h^{q,p}$ also be given. The $h^{p,q}$ are called the Hodge numbers. We define the period domain D to be the set of polarized Hodge structures (V, Q, φ) with $\dim(V^{p,q}) = h^{p,q}$. Therefore, each Hodge structure satisfies both HR-relations for Q. The period domain is a homogeneous space. If we fix a Hodge structure φ_0 with isotropy group H_0 in $G(\mathbb{R})$, then $D = G(\mathbb{R})/H_0$. *For all the examples we consider, there exists a CM Hodge structure in D. Therefore we may, and we will, assume that φ_0 is a fixed CM Hodge structure.* We have a bijection (with g ranging over $G(\mathbb{R})$),

$$\left\{ g\varphi_0 g^{-1} = \varphi_g : \mathbb{U}(\mathbb{R}) \to G(\mathbb{R}) \right\} \simeq G(\mathbb{R})/H_0$$
$$g\varphi_0 g^{-1} \to gH_0.$$

In order to introduce the analogue of Schneider's Theorem, we need the context of variations of Hodge structure since, in general, there may not exist suitable $G(\mathbb{Z})$–invariant functions on D. From now on, we do not use the abstract setting, as our examples are geometric. Indeed, all the examples we consider in this paper, and in [22], are smooth proper algebraic families defined over $\overline{\mathbb{Q}}$:

$$\pi : \mathcal{X} \to S.$$

In particular, the map π is surjective and proper. The base S is a quasi-projective variety defined over $\overline{\mathbb{Q}}$. Moreover, the fibers $\mathcal{X}_s$, $s \in S$, are smooth projective varieties, with $\mathcal{X}_s(\mathbb{C})$ a compact Kähler n-fold. When $s \in S(\overline{\mathbb{Q}})$, the fiber $\pi^{-1}(s) = \mathcal{X}_s$ is defined over $\overline{\mathbb{Q}}$ as an algebraic variety. Let o be a fixed base point in S and let $V = H^n(\mathcal{X}_o, \mathbb{Q})_{\text{prim}}$, the primitive cohomology, with its usual polarization Q (see [15], p.14, or [26]), given by

$$Q(v, w) = \int_{\mathcal{X}_o} v \wedge w. \tag{2}$$

When X is a curve, or a Calabi–Yau 3-fold, we have $H^n(X, \mathbb{Q})_{\text{prim}} = H^n(X, \mathbb{Q})$, $n = \dim X$.

For $s \in S$, the filtration associated to the usual Hodge decomposition, namely $H^n(\mathcal{X}_s, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(\mathcal{X}_s)$, can be pulled back to a filtration of $V_{\mathbb{C}}$ with Hodge numbers independent of s. We denote either by $\varphi_{\mathcal{X}_s}$ or by $H^n(\mathcal{X}_s, \mathbb{Q}_{\mathcal{X}_s})$ the corresponding Hodge structure on V. The induced map from S to the corresponding period domain D is multi-valued when S has non-trivial fundamental group, but its image in $\Gamma\backslash D$ is well-defined, where $\Gamma \subseteq G(\mathbb{Z})$ is the image of the monodromy representation ([5], Chapter 4, [25], Chapter 1). Therefore, we have a well-defined period map

$$\Phi : S \to \Gamma\backslash D.$$

Let $\rho : D \to \Gamma\backslash D$ be the natural projection. We can now state the analogue of Schneider's problem on the j-function in this context (it is a special case of Conjecture (VIII.A.8) of [7]).

Problem: Let $s \in S(\overline{\mathbb{Q}})$ and suppose that $\varphi \in D$ satisfies $\rho(\varphi) = \Phi(s)$. Show that $\varphi = g\varphi_o g^{-1}$ for $g \in G(\overline{\mathbb{Q}})$ if and only if (V, φ) has CM.

When the pair (V, Q) is clear from the context we just say "φ is conjugate over $\overline{\mathbb{Q}}$ to a CM Hodge structure" instead of "$\varphi = g\varphi_o g^{-1}$ for $g \in G(\overline{\mathbb{Q}})$" .

The "if" part of the above statement is immediate in the examples we consider, the only work being in the "only if" part. Notice that, once one choice of $\varphi \in D$ with $\rho(\varphi) = \Phi(s)$ is conjugate in $G(\overline{\mathbb{Q}})$ to φ_o, then every $\varphi \in D$ with $\rho(\varphi) = \Phi(s)$ is conjugate in $G(\overline{\mathbb{Q}})$ to φ_o.

Using either the description in terms of complex structures on $\mathbb{R}^{2g}$, or the real model, of the Siegel upper half space $\mathcal{H}_g$ of genus g (see [16], §3), we have:

Proposition: When $\pi : \mathcal{X} \to S$ is a family of smooth projective algebraic curves of genus g, with the above assumptions, we may take $D = \mathcal{H}_g$ and $\Gamma \subseteq \mathrm{PSp}(2g, \mathbb{Z})$. Then, the statement of the Problem is true by [4], [17], [19].

In [22] we show the following:

Claim: The statement of the Problem is true for all the families of Calabi–Yau manifolds with dense sets of CM fibers constructed by Rohde in [15] (and called CMCY families in that same reference).

In this paper, we focus on two examples of families of Calabi–Yau 3-folds with dense sets of CM fibers, studied respectively by Borcea and Viehweg–Zuo, and the first step in a tower of Calabi–Yau manifolds that starts with these two examples. We use the fact that the Hodge structures associated to each fiber of our families are sub-Hodge structures of ones involving direct sums and tensor products of Hodge structures on curves, and various CM Hodge structures. The CM criterion on a curve is then the one from [4], [19]. Similar considerations allow one to deal with all the examples of [15]. Indeed, this is directly related to the proofs that these families have dense sets of CM fibers. The definition of a CMCY family in [15], Chapter 7, p.143, involves a stronger CM condition. Namely, a family of Calabi–Yau n-manifolds over a quasi-projective base space, which contains a Zariski dense set of fibers X such that the Mumford–Tate group of $H^k(X, \mathbb{Q}_X)$ is a torus for all k, is defined to be a CMCY family. All the examples we consider satisfy the stronger CMCY condition. We say that a variety X (Calabi–Yau or not) such that the Mumford–Tate group of $H^k(X, \mathbb{Q}_X)$ is a torus for all k "has CM for all levels".

3 The main lemmas

In this section we collect, for the convenience of the reader, the main lemmas that we use from other references.

Lemma 1: [2],[23] (i) Let (V_1, φ_1) and (V_2, φ_2) be two Hodge structures of weight n and $\varphi_1 \oplus \varphi_2$ the induced Hodge structure on $V_1 \oplus V_2$. Then,

$$M_{\varphi_1 \oplus \varphi_2} \subset M_{\varphi_1} \times M_{\varphi_2} \subset \mathrm{SL}(V_1) \times \mathrm{SL}(V_2) \subset \mathrm{SL}(V_1 \oplus V_2),$$

and the projections

$$M_{\varphi_1 \oplus \varphi_2} \to M_{\varphi_1}, \qquad M_{\varphi_1 \oplus \varphi_2} \to M_{\varphi_2}$$

are surjective.

(ii) The Mumford-Tate group does not change under Tate twists.

(iii) The Mumford-Tate group of a Hodge structure concentrated in bidegree (p,p), $p \in \mathbb{Z}$, is trivial.

(iv) Let $\varphi_1 \otimes \varphi_2$ be the induced Hodge structure on $V_1 \otimes V_2$. Then $\varphi_1 \otimes \varphi_2$ has CM if and only if both φ_1 and φ_2 have CM.

Lemma 2: [15], [26]. Let X_1 and X_2 be compact Kähler manifolds. Then, for any integers $k, r, s \geq 0$ we have

$$H^k(X_1 \times X_2, \mathbb{Q}) = \oplus_{i+j=k} H^i(X_1, \mathbb{Q}) \otimes H^j(X_2, \mathbb{Q})$$

and

$$H^{r,s}(X_1 \times X_2) = \oplus_{p+p'=r,\, q+q'=s} H^{p,q}(X_1) \otimes H^{p',q'}(X_2).$$

Lemma 3: [15], [26]. Let X be an algebraic manifold of dimension n and let $\widehat{X}$ be the blow-up of X along a submanifold Z of codimension 2 in X. Then, for all k, we have an isomorphism of Hodge structures

$$H^k(X, \mathbb{Q}_X) \oplus H^{k-2}(Z, \mathbb{Q}_Z)(-1) \simeq H^k(\widehat{X}, \mathbb{Q}_{\widehat{X}}),$$

where $H^{k-2}(Z, \mathbb{Q}_Z)(-1)$ is $H^{k-2}(Z, \mathbb{Q}_Z)$ shifted by $(1,1)$ in bidegree. Therefore, the Mumford–Tate group of $H^k(\widehat{X}, \mathbb{Q}_{\widehat{X}})$ is commutative if and only if the Mumford–Tate groups of both $H^k(X, \mathbb{Q}_X)$ and $H^{k-2}(Z, \mathbb{Q}_Z)$ are commutative. What's more, if X is a smooth surface and Z is a point of X, then the Mumford–Tate groups of $H^2(X, \mathbb{Q}_X)$ and $H^2(\widehat{X}, \mathbb{Q}_{\widehat{X}})$ are isomorphic.

4 The Borcea family as a 2-step tower

Let

$$M_1 = \{x = (x_i)_{i=1}^4 \in \mathbb{P}_1^4 : x_i \neq x_j, i \neq j\}/\mathrm{Aut}(\mathbb{P}_1),$$

where $\mathrm{Aut}(\mathbb{P}_1)$ acts diagonally. It is non-canonically isomorphic to

$$\Lambda = \mathbb{P}_1 \setminus \{0, 1, \infty\}.$$

Consider three families $\mathcal{E}_i$, $i = 1, 2, 3$, of elliptic curves (Calabi–Yau 1-folds)

$$\mathcal{E}_i \to \Lambda$$

with fiber $\mathcal{E}_{\lambda_i}$ of $\mathcal{E}_i$ at $\lambda_i \in \Lambda$ given by

$$y^2 = x(x-1)(x-\lambda_i), \qquad i = 1, 2, 3.$$

By the theorem of Th. Schneider [17] mentioned in §1, the statement of the Problem in §2 is true for these families of elliptic curves.

Each elliptic curve $\mathcal{E}_{\lambda_i}$ carries an involution $\iota_i : (x, y) \mapsto (x, -y)$ fixing the group $\mathcal{E}_{\lambda_i}[2]$ of 2-torsion points, which has 4 elements, and reversing the sign of the holomorphic 1-form dx/y. The product involution $\iota = \iota_2 \times \iota_3$ on the abelian surface $\mathcal{A}_{\lambda_2,\lambda_3} = \mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}$ sends a point to its group inverse and has $4 \times 4 = 16$ fixed points. Blowing up these 16 points, we get a surface $\widehat{\mathcal{A}}_{\lambda_2,\lambda_3}$ with an involution $\widehat{\iota}$, induced by ι, whose ramification locus is the 16 exceptional divisors. The quotient $\mathcal{K}_{\lambda_2,\lambda_3} = \widehat{\mathcal{A}}_{\lambda_2,\lambda_3}/\widehat{\iota}$ is smooth and is a $K3$-surface (hence a Calabi–Yau 2-fold), called the Kummer surface of $\mathcal{A}_{\lambda_2,\lambda_3}$. The surface $\mathcal{K}_{\lambda_2,\lambda_3}$ is isomorphic to one obtained by resolving the 16 singular double points of the quotient $\mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}/\iota_2 \times \iota_3$. On this last quotient surface, the maps $\iota_2 \times \mathrm{Id}$ and $\mathrm{Id} \times \iota_3$ define the same involution, which in turn induces an involution σ on $\mathcal{K}_{\lambda_2,\lambda_3}$. The involutions $\widehat{\iota}$ and σ exist by the universal property of blowing-up (see [9], II, Corollary 7.15). The ramification locus R_σ of σ has 8 connected components consisting of smooth rational curves given by the union of the image, under the degree 2 rational map $\mathcal{A}_{\lambda_2,\lambda_3} \to \mathcal{K}_{\lambda_2,\lambda_3}$, of $\mathcal{E}_{\lambda_2}[2] \times \mathcal{E}_{\lambda_3}$ and of $\mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}[2]$. The involution σ reverses the sign of any (non-zero) holomorphic 2-form on $\mathcal{K}_{\lambda_2,\lambda_3}$. This construction is a first step in a tower: we build a Calabi–Yau 2-fold with involution reversing the sign of any holomorphic 2-form, from two Calabi–Yau 1-folds with involution reversing the sign of any holomorphic 1-form. What's more, the rational Hodge structure $\varphi_{\lambda_2,\lambda_3}$ of level 2 on $\mathcal{K}_{\lambda_2,\lambda_3}$ is the

$\iota_2 \times \iota_3$–invariant part of the weight 2 Hodge structure on $\mathcal{A}_{\lambda_2,\lambda_3}$. This is just the tensor product of the rational Hodge structure φ_{λ_2} of level 1 on $\mathcal{E}_{\lambda_2}$ with that, φ_{λ_3}, on $\mathcal{E}_{\lambda_3}$. This is a CM Hodge structure if and only if both $\mathcal{E}_{\lambda_2}$ and $\mathcal{E}_{\lambda_3}$ have CM by [2], Proposition 1.2 (Lemma 1(iv), §3). Suppose $\varphi_{\lambda_2,\lambda_3}$ is conjugate over $\overline{\mathbb{Q}}$ to a CM Hodge structure φ_o. We can write φ_o as a tensor product $\varphi_{0,2} \otimes \varphi_{0,3}$ of weight 1 CM Hodge structures on the elliptic curves. By [10], Proposition 2.5, p.563, it follows that φ_{λ_2} and φ_{λ_3} are both also conjugate over $\overline{\mathbb{Q}}$ to a CM Hodge structure. Applying Th. Schneider's theorem [17], we deduce that $\mathcal{E}_{\lambda_2}$ and $\mathcal{E}_{\lambda_3}$ are both CM and hence that $\varphi_{\lambda_2,\lambda_3}$ is CM. We therefore have:

Theorem 1: The statement of the Problem of §2 holds for the family

$$\mathcal{K} \to \Lambda^2$$

of Calabi–Yau 2-folds constructed above, which has a dense set of CM fibers.

The next step in the tower applies a construction similar to the above, but now to $(\mathcal{E}_1, \iota_1)$ and $(\mathcal{K}, \sigma)$ (see [3]). Let $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^3$, and blow up the product $\mathcal{T}_{\lambda_1,\lambda_2,\lambda_3} = \mathcal{E}_{\lambda_1} \times \mathcal{K}_{\lambda_2,\lambda_3}$ along the connected components of the codimension 2 ramification divisor $\mathcal{E}_{\lambda_1}[2] \times R_\sigma$ of the involution $\iota_1 \times \sigma$. Consider the induced involution $\widehat{\iota_1 \times \sigma}$ on this blow-up $\widehat{\mathcal{T}_{\lambda_1,\lambda_2,\lambda_3}}$. The quotient

$$\mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3} = \widehat{\mathcal{T}_{\lambda_1,\lambda_2,\lambda_3}}/\widehat{\iota_1 \times \sigma}$$

is a Calabi–Yau 3-fold with involution γ induced by $\mathrm{Id} \times \sigma = \iota_1 \times \mathrm{Id}$ on $\mathcal{E}_{\lambda_1} \times \mathcal{K}_{\lambda_1,\lambda_2}/\iota_1 \times \sigma$ of which $\mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3}$ is a resolution. It is also a minimal resolution of

$$\mathcal{E}_{\lambda_1} \times \mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}/H$$

where H is the group of order 4 generated by $\iota_1 \times \iota_2 \times \mathrm{Id}$ and $\mathrm{Id} \times \iota_2 \times \iota_3$. The singularities of this last quotient lie along a configuration of 48 rational curves with 4^3 intersection points. The ramification locus R_γ of γ consists of the image under the degree 4 rational map

$$\mathcal{E}_{\lambda_1} \times \mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3} \to \mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3}$$

of the union of

$$\mathcal{E}_{\lambda_1}[2] \times \mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}, \quad \mathcal{E}_{\lambda_1} \times \mathcal{E}_{\lambda_2}[2] \times \mathcal{E}_{\lambda_3}, \quad \mathcal{E}_{\lambda_1} \times \mathcal{E}_{\lambda_2} \times \mathcal{E}_{\lambda_3}[2].$$

Moreover γ reverses the sign of any holomorphic 3-form on $\mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3}$. The Hodge structure $\varphi_{\lambda_1,\lambda_2,\lambda_3}$ given by $H^3(\mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3}, \mathbb{Q}_{\mathcal{Y}_{\lambda_1,\lambda_2,\lambda_3}})$ is $\varphi_{\lambda_1} \otimes \varphi_{\lambda_2} \otimes \varphi_{\lambda_3}$, where φ_{λ_i} is the level 1 rational Hodge structure of the elliptic curve $\mathcal{E}_{\lambda_i}$ (for details see [2]). Therefore, again by Th. Schneider's theorem [17], and the fact that $\varphi_{\lambda_1} \otimes \varphi_{\lambda_2} \otimes \varphi_{\lambda_3}$ is CM if and only if each φ_i, $i = 1, 2, 3$ is CM, we deduce easily that

Theorem 2: The statement of the Problem of §2 holds for the family

$$\mathcal{Y} \to \Lambda^3$$

of Calabi–Yau 3-folds constructed above, which has a dense set of CM fibers.

5 The Viehweg–Zuo family

Viehweg and Zuo [23] have constructed iterated cyclic covers of degree 5 which give a family of Calabi-Yau 3-folds (which we call the VZCY family) with a dense set of CM fibers. The fibers of the family are smooth quintics in $\mathbb{P}_4$. The study of this family is taken up again in [15], §7.3. For a projective hypersurface $X \subset \mathbb{P}_4$, only the Mumford-Tate group of the Hodge structure on $H^3(X, \mathbb{Q})$ can be non-trivial, so the CM condition in the CMCY definition is just the usual one. Consider the parameter space

$$M_2 = \{(x_i)_{i=1}^5 \in \mathbb{P}_1^5 : x_i \neq x_j, \quad i \neq j\}/\mathrm{Aut}(\mathbb{P}_1)$$

which is non-canonically isomorphic to

$$S = \{u, v \in \mathbb{P}_1(\mathbb{C}) : u \neq 0, 1, \infty, \quad v \neq 0, 1, \infty, \quad u \neq v\}.$$

Explicitly, the VZCY family is given by

$$\pi : \mathcal{X} \to S$$

with fiber $\mathcal{X}_{(u,v)}$ the projective variety with equation,

$$x_4^5 + x_3^5 + x_2^5 + x_1(x_1 - x_0)(x_1 - ux_0)(x_1 - vx_0)x_0 = 0, \tag{3}$$

in homogeneous coordinates $[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}_4$. The fibers $\mathcal{X}_{(u,v)}$ are smooth hypersurfaces of degree 5 in $\mathbb{P}_4$. They are therefore Calabi–Yau

3–folds, by the well known fact that any smooth hypersurface of degree $d+1$ in $\mathbb{P}_d$ is a Calabi-Yau $(d-1)$-fold. As in §2, fix a base point $o \in S$ and let $V = H^3(\mathcal{X}_o, \mathbb{Q})$. The VZCY family is an example of an iterated cyclic cover. Indeed, consider the family of smooth algebraic curves of genus 6 in $\mathbb{P}_2$ given by the following family $\mathcal{C} \to S$ of cyclic covers of $\mathbb{P}_1$ of degree 5:

$$x_2^5 + x_1(x_1 - x_0)(x_1 - ux_0)(x_1 - vx_0)x_0 = 0, \qquad (u,v) \in S. \tag{4}$$

The fibers of this family are the ramification loci of the family $\mathcal{S} \to S$ of cyclic covers of $\mathbb{P}_2$ of degree 5 given by the family of smooth surfaces in $\mathbb{P}_3$:

$$x_3^5 + x_2^5 + x_1(x_1 - x_0)(x_1 - ux_0)(x_1 - vx_0)x_0 = 0. \tag{5}$$

Iterating again, the fibers of this last family are the ramification loci of the family of cyclic covers of $\mathbb{P}_3$ of degree 5 given by the VZCY family.

Let $\mathcal{F}_5$ be the Fermat curve of degree 5 given by $x^5+y^5+z^5 = 0$. The usual Hodge structure $(H^1(\mathcal{F}_5, \mathbb{Q}), \varphi_{\mathcal{F}_5})$ associated to the Hodge decomposition $H^1(\mathcal{F}_5, \mathbb{C}) = H^{(1,0)}(\mathcal{F}_5) \oplus H^{(0,1)}(\mathcal{F}_5)$ has CM, since it is well known that the Jacobian of every Fermat curve is of CM type.

Let $s = (u,v) \in S$, with $u, v \in \overline{\mathbb{Q}}$. Suppose, in addition, that the usual Hodge decomposition on $H^3(\mathcal{X}_s, \mathbb{C})$ gives a representative homomorphism

$$\varphi_s : \mathbb{U}(\mathbb{R}) \to \mathrm{SL}(V)_{\mathbb{R}}$$

satisfying $\varphi_s = g\varphi_o g^{-1}$ for $g \in G(\overline{\mathbb{Q}}) \subseteq \mathrm{SL}(V)_{\overline{\mathbb{Q}}}$, where $G = \mathrm{Aut}(V, Q)$ with Q as in §2, (2), and φ_o is a fixed CM Hodge structure.

By the argument following Claim 8.6 of [23], p.525, the Hodge structure $(H^3(\mathcal{X}_o, \mathbb{Q}), \varphi_s)$ is a sub-Hodge structure of the Hodge structure given by

$$[\varphi_s^1 \otimes \varphi_{\mathcal{F}_5} \otimes \varphi_{\mathcal{F}_5}] \oplus [\varphi_{\mathcal{F}_5} \otimes \mathrm{Id}_W] \oplus [\varphi_s^1(-1)]$$

on

$$[H^1(\mathcal{C}_o, \mathbb{Q}) \otimes H^1(\mathcal{F}_5, \mathbb{Q}) \otimes H^1(\mathcal{F}_5, \mathbb{Q})] \oplus [H^1(\mathcal{F}_5, \mathbb{Q}) \otimes W] \oplus [H^1(\mathcal{C}_o, \mathbb{Q})(-1)],$$

where (-1) denotes the Tate twist and W is a $\mathbb{Q}$-vector space with a constant $(1,1)$-Hodge structure. For each $s \in S$, the homomorphism φ_s^1 is associated to the usual Hodge decomposition $H^1(\mathcal{C}_s, \mathbb{C}) = H^{(1,0)}(\mathcal{C}_s) \oplus H^{(0,1)}(\mathcal{C}_s)$. It is now easy to see that, if $\varphi_s = g\varphi_o g^{-1}$ for $g \in G(\overline{\mathbb{Q}}) \subseteq \mathrm{SL}(V)_{\overline{\mathbb{Q}}}$, then we have $\varphi_s^1 = h\varphi_1 h^{-1}$ for $(H^1(\mathcal{C}_o, \mathbb{Q}), \varphi_1)$ CM and $h \in \mathrm{Sp}(12, \overline{\mathbb{Q}})$. Therefore, by the

Proposition of §2, we have that φ_s^1 is CM. Now, as φ_s is therefore a sub-Hodge structure of a Hodge structure built up of tensor products and direct sums of CM Hodge structures, by Lemma 8.1 of [23] (see also the lemmas of our §3), it follows that φ_s has CM as required. We therefore have

Theorem 3: The statement of the Problem of §2 holds for the VZCY family of Calabi–Yau 3-folds constructed above, which has a dense set of CM fibers.

On each fiber $\mathcal{X}_s$, $s \in S$, we have the involution

$$\delta : ([x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_0 : x_1 : x_2 : x_4 : x_3]$$

which leaves the smooth divisor $D_s : x_3 = x_4$ invariant. Moreover D_s is isomorphic to $\mathcal{S}_s$ of (5), which is CMCY for a dense set of $s \in S$ (see [15], p.151). Moreover, by [23], p.525, $H^2(\mathcal{S}_s, \mathbb{Q}_{\mathcal{S}_s})$ is a sub-Hodge structure of the tensor product of $H^1(\mathcal{C}_s, \mathbb{Q}_{\mathcal{C}_s})$ and $H^1(\mathcal{F}_5, \mathbb{Q}_{\mathcal{F}_5})$, so, using arguments similar to the above, the statement of the Problem of §2 holds for the family $\mathcal{S} \to S$.

The fibers of the VZCY family isomorphic to the Fermat quintic three-fold have CM (see [23],[15], p.151). The periods of the holomorphic 3-forms defined over $\overline{\mathbb{Q}}$, and their transcendence, is discuss in the Appendix, authored by Marvin D. Tretkoff.

6 The first step of a Borcea–Voisin tower

In this section we indicate how to prove the Claim of §2 for the Borcea–Voisin towers of CMCY manifolds constructed by Rohde [15], by summarizing the ideas for one step in such a tower using the families of §4 and §5. In §4, we already saw examples of such a construction. Full details for the general case will be given in [22].

Using the CMCY families with involution of §4, §5, we build a CMCY family with involution of higher dimension using the construction in [3] and in [15], Proposition 7.2.5, and show that the statement of the Problem of §2 holds for this new family.

Let $(\mathcal{Y}, \gamma)$ be the Borcea family of Calabi–Yau 3-folds with involution constructed in §4, and $(\mathcal{X}, \delta)$ be the VZCY family of Calabi–Yau 3-folds with involution from §5. Let $\mathcal{Y}_{1,2,3}$ be the fiber of $\mathcal{Y}$ at $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^3$ and $\mathcal{X}_s$ be the fiber of $\mathcal{X}$ at $s \in S$. The ramification divisors $R_\gamma = D_{1,2,3} \subset \mathcal{Y}_{1,2,3}$ of γ and $D_s \subset \mathcal{X}_s$ of δ consist of smooth non-trivial disjoint hypersurfaces.

Indeed, from §4, the divisor $D_{1,2,3}$ is CM for all levels, as $\mathbb{P}_1$ carries the trivial CM Hodge structure for all levels. As noted at the end of §5, divisor D_s is isomorphic to $\mathcal{S}_s$ of (5) for which the statement of the Problem of §2 holds for all levels.

Let $\widehat{\mathcal{Y}_{1,2,3} \times \mathcal{X}_s}$ be the blow-up of $\mathcal{Y}_{1,2,3} \times \mathcal{X}_s$ with respect to $D_{1,2,3} \times D_s$, and $\widehat{\gamma \times \delta}$ be the involution on the blow-up induced by $\gamma \times \delta$. Then

$$\mathcal{Z}_{1,2,3,s} = \widehat{\mathcal{Y}_{1,2,3} \times \mathcal{X}_s}/\widehat{\gamma \times \delta}$$

is a Calabi–Yau 6-fold with involution ε generating $(\mathbb{Z}/2 \times \mathbb{Z}/2)/\langle \gamma \times \delta \rangle$, whose fixed points are a smooth non-trivial divisor, and which reverses the sign of any holomorphic 6-form on $\mathcal{Z}_{1,2,3,s}$ [3], [15], [24].

Following [15], Chapter 7, and [26], using Lemmas 1,2,3 of §3, we express the Hodge structure on $H^*(\mathcal{Z}_{1,2,3,s}, \mathbb{Q})$, in terms of tensor products and direct sums of the Hodge structures on $H^*(\mathcal{Y}_{1,2,3}, \mathbb{Q})$, $H^*(\mathcal{X}_s, \mathbb{Q})$, $H^*(D_{1,2,3}, \mathbb{Q})$ and $H^*(D_s, \mathbb{Q})$, where several levels $*$ may intervene. By using this analysis of the Hodge structure, we deduce that the statement of the Problem of §2 holds for the family $\mathcal{Z}$, since it holds for the family $\mathcal{Y}$ of §4 (Theorem 2), for the family $\mathcal{X}$ of §5 (Theorem 3), and for the ramification divisors. The fact that several levels of Hodge structure are involved is not an obstacle, as all relevant fibers of our families turn out to have CM for all levels.

For example, the Hodge structure of level 6 on $\mathcal{Z}_{1,2,3,s}$ is given by the Hodge sub–structure of level 6 on $\widehat{\mathcal{Y}_{1,2,3} \times \mathcal{X}_s}$ invariant under $\widehat{\gamma \times \delta}$. Using Lemma 3 of §3, this, in turn, can be expressed in terms of the Hodge structure of level 6 on $\mathcal{Y}_{1,2,3} \times \mathcal{X}_s$ and the Hodge structure of level 4 on $D_{1,2,3} \times D_s$. We then use Lemmas 1,2 of §3 to express these Hodge structures in terms of tensor products and direct sums of the Hodge structures, of various levels, on $\mathcal{Y}_{1,2,3}$, $\mathcal{X}_s$, $D_{1,2,3}$, D_s. If the Hodge structure $H^6(\mathcal{Z}_{1,2,3,s}, \mathbb{Q}_{\mathcal{Z}_{1,2,3,s}})$ is conjugate over $\overline{\mathbb{Q}}$ to a CM Hodge structure, the same is true of the Hodge structures on $\mathcal{Y}_{1,2,3}$, $\mathcal{X}_s$, $D_{1,2,3}$, D_s. As the statement of the Problem of §2 applies to them, again using Lemmas 1,2,3 of §3, we deduce that the statement of the Problem of §2 holds for the family $\mathcal{Z} \to \Lambda^3 \times S$.

Appendix: transcendence of the periods on Calabi-Yau-Fermat hypersurfaces

Marvin D. Tretkoff

Dedicated to the memory of Leon Ehrenpreis, my Ph.D. advisor and friend for 50 years.

A famous transcendence theorem of Th. Schneider (see Schneider [18], Siegel [20]) asserts that if ω is a holomorphic 1-form on a compact Riemann surface of genus at least 1, then there is a 1-cycle γ on that Riemann surface such that $\int_\gamma \omega$ is a transcendental number. Here, the Riemann surface and the 1-form ω are both supposed to be defined over the same algebraic number field. The possibility of generalizing Schneider's theorem to higher dimensional hypersurfaces is a natural question.

Let V denote the Fermat hypersurface defined in affine coordinates by the equation

$$z_1^r + ... + z_{n+1}^r = 1.$$

In [21], we explicitly determined the periods of the n-forms on V for all values of n and r. When $r = n+2$, V is a Calabi-Yau manifold, because on it there is a nowhere vanishing holomorphic n-form, ω, given by

$$\omega = z_{n+1}^{-(n+1)} dz_1 ... dz_n.$$

In order that Schneider's theorem generalize to these Calabi-Yau manifolds it is necessary and sufficient that at least one period of ω be transcendental.

For these hypersurfaces, the periods of ω are

$$\int_\gamma \omega = \alpha(\gamma)\ \Gamma(1/(n+2))^{n+1}/\Gamma((n+1)/(n+2)),$$

where $\Gamma(u)$ is the classical gamma function applied to u. Here γ can be any n-cycle on V that is not null-homologous and $\alpha(\gamma)$ is an *algebraic number that depends on γ*. It follows that we have the

Theorem: Schneider's Theorem extends to the n-dimensional Fermat hypersurfaces of degree $n+2$ if and only if

$$(*) \qquad \Gamma(1/(n+2))^{n+1}/\Gamma((n+1)/(n+2))$$

is a transcendental number.

Here, we provide some details for the Fermat quintic three-fold, V, defined in affine coordinates by the equation

$$x^5 + y^5 + z^5 + w^5 = 1.$$

A nowhere vanishing holomorphic 3-form on V is given by

$$\omega = w^{-4} dx dy dz.$$

Now, let

$$A(x,y,z,w) = (\zeta x, y, z, w), \quad B(x,y,z,w) = (x, \zeta y, z, w),$$

$$C(x,y,z,w) = (x, y, \zeta z, w), \quad D(x,y,z,w) = (x, y, z, \zeta w),$$

with ζ a primitive 5th root of unity, be automorphisms of the ambient 4-space. Clearly, V is left fixed by A, B, C, D and, therefore, also by the group ring $\mathbb{Z}[A,B,C,D]$. It is shown in [21] that there is a primitive 3-cycle Γ on V for which we have the following result.

Theorem: (a) The images

$$\Gamma(i,j,k,\ell) = A^{(i-1)} B^{(j-1)} C^{(k-1)} D^{(\ell-1)} \Gamma$$

span a cyclic $\mathbb{Z}[A,B,C,D]$-module and a subset of them forms a basis for the group of primitive 3-cycles on V. Here, we recall that a 3-cycle on V is termed "primitive" if it is not contained in any hyperplane section.

(b) The 3-form ω can be evaluated explicitly along the $\Gamma(i,j,k,\ell)$. In fact,

$$\int_{\Gamma(i,j,k,\ell)} \omega = \frac{1}{5^3} \zeta^{i+j+k+\ell} (1-\zeta)^4 \Gamma(1/5)^4 \Gamma(4/5)^{-1}.$$

Therefore, each period of ω is the product of an algebraic number and $\Gamma(1/5)^4\Gamma(4/5)^{-1}$. The algebraic number depends on the 3-cycle in question.

It follows that Schneider's theorem generalizes to the Fermat quintic threefold if and only if $\Gamma(1/5)^4\Gamma(4/5)^{-1}$ *is transcendental.*

Apparently the transcendence of this number is unknown.

A similar result is valid for all the higher dimensional Calabi-Yau-Fermat manifolds. In these cases, the number $\Gamma(1/5)^4\Gamma(4/5)^{-1}$ will be replaced by another number, $(*)$, involving values of the gamma function.

Finally, we note that the formula for the periods of n-forms on Fermat hypersurfaces of degree $r \neq n+2$ (see [21]) is substantially more complicated than that for the Calabi-Yau-Fermat hypersurfaces treated in the present note.

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