

M147 Practice Problems for Exam 1

Exam 1 will cover sections 1.1, 1.2, 1.3, 3.1, 3.2, 3.3, 3.4, 3.5, 4.1, and 4.2. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit.

1. Which of the following functions corresponds with the graph in Figure 1.

- (a) $y = \log_2 x$
- (b) $y = \log_{\frac{1}{2}} x$
- (c) $y = 2^x$
- (d) $y = \left(\frac{1}{2}\right)^x$

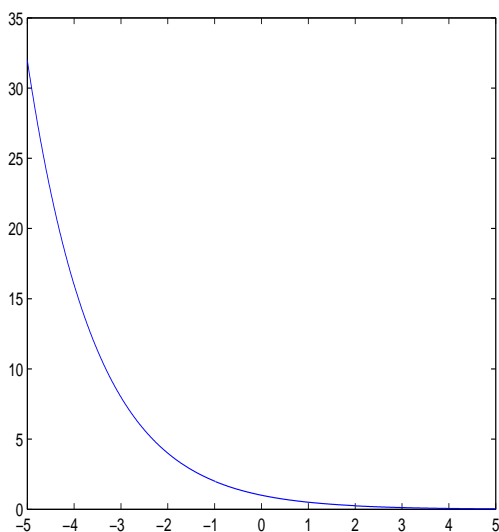


Figure 1: Figure for Problem 1.

2. Use a logarithmic transformation to find a linear relationship between (appropriate transformations of) x and y if

$$y = 2 \times 7^{4x}.$$

3. Given the semilog plot in Figure 2, find a functional relationship between x and y .

4. Given the double-log plot in Figure 3, find a functional relationship between x and y .

5. Compute each of the following limits:

5a.

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}.$$

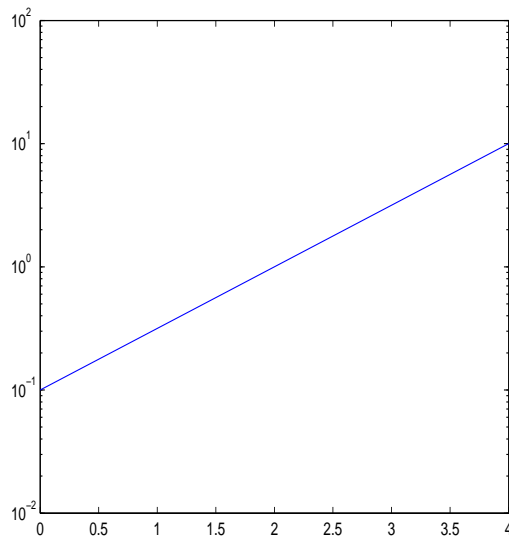


Figure 2: Figure for Problem 3.

5b.

$$\lim_{x \rightarrow 3^-} \frac{x}{x^2 - 2x - 3}.$$

5c.

$$\lim_{x \rightarrow 0} \frac{\sin 7x}{x}.$$

5d.

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1}.$$

5e.

$$\lim_{x \rightarrow -\infty} \frac{x^3 - x^2 + 1}{1 - x^2}.$$

5f.

$$\lim_{x \rightarrow \infty} (e^{-x} \sin x).$$

6. Find all points at which

$$\frac{\ln(1 - x)}{\ln(1 + x)}$$

is continuous.

7. Find a value for c that makes the given function continuous at all points.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ x - c, & x > 1 \end{cases}.$$

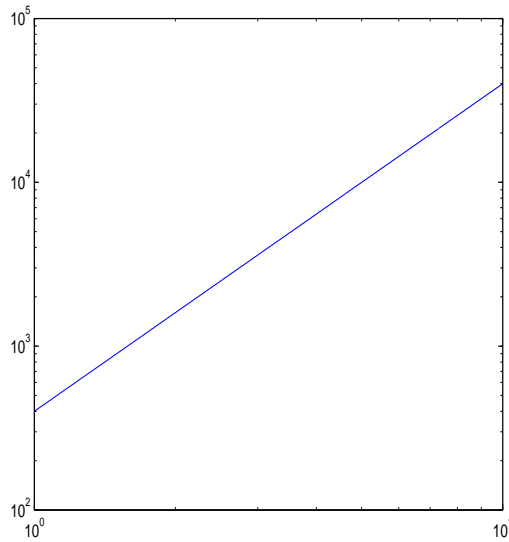


Figure 3: Figure for Problem 4.

8. Prove that the equation

$$e^x - 2 = \sin x$$

has at least one real-valued solution.

9. Use the bisection method to approximate a root of

$$x^4 + x^3 + x - 1 = 0$$

with a maximum error less than $\frac{1}{3}$.

10. Use the bisection method to approximate $\sqrt[4]{5}$ with a maximum error less than $\frac{1}{3}$.

11. Use the definition of derivative to compute the derivative of the following function at $x = 0$.

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

12. Determine whether or not each of the following functions is differentiable at the point $x = 0$. In each case, explain why or why not.

12a.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ x^2 - 1, & x > 0 \end{cases}.$$

12b.

$$f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ 2x + 1, & x > 0 \end{cases}.$$

12c.

$$f(x) = x|x|.$$

13. Find an equation for the line that is tangent to the given curve at $x = 1$.

$$y = x^3 + 1.$$

Sketch a graph of the curve along with this tangent line.

14. A car moves along a straight road. Its location at time t is given by

$$s(t) = 20t^2, \quad 0 \leq t \leq 2,$$

where t is measured in hours and $s(t)$ is measured in kilometers.

14a. Graph $s(t)$ for $0 \leq t \leq 2$.

14b. Find the average velocity of the car between $t = 0$ and $t = 2$. Illustrate the average velocity on the graph of $s(t)$.

14c. Find the instantaneous velocity of the car at $t = 1$. Illustrate the instantaneous velocity on the graph of $s(t)$.

15. Find a point on the curve

$$y = x^2 + x + 1$$

whose tangent line is parallel to the line

$$y - 2 = 3(x - 1).$$

Solutions

1. Notice that the function is defined for negative values of x , so it cannot be either of the logarithms. Since the function is decreasing, it must be exponentiation with a base less than 1, and this leaves only (d) $y = (\frac{1}{2})^x$.

2. You can proceed by taking a logarithm of this equation to any base. Though 7 would be a reasonable base here, it is sufficiently uncommon that I'll use base 10. That is,

$$\log y = \log 2 \times 7^{4x} \Rightarrow \log y = \log 2 + 4x \log 7.$$

The linear relationship is

$$\log y = (4 \log 7)x + \log 2.$$

3. This is a semilog plot with $\log y$ on the vertical axis and x on the horizontal. That is, the line has an equation of the form

$$\log y = mx + b.$$

We can read directly from the plot that $b = -1$. (I.e., $b = \log 10^{-1} = -1$) Likewise, the slope is

$$m = \frac{\log 10^1 - \log 10^{-1}}{4 - 0} = \frac{1 - (-1)}{4} = \frac{1}{2}.$$

We have, then,

$$\log y = \frac{1}{2}x - 1.$$

In order to get a functional relationship, exponentiate each side with base 10,

$$10^{\log y} = 10^{\frac{1}{2}x - 1} \Rightarrow y = 10^{-1}(10^{\frac{1}{2}})^x.$$

4. This is a double-log plot, so we look for a relationship of the form

$$\log y = m \log x + b,$$

for which

$$y = 10^b x^m.$$

Reading the plot, we see that

$$m = \frac{\log(4 \times 10^4) - \log(4 \times 10^2)}{\log 10^1 - \log 10^0} = \frac{\log \frac{4 \times 10^4}{4 \times 10^2}}{1} = \log 10^2 = 2.$$

Likewise,

$$b = \log 400.$$

We conclude

$$y = 10^{\log 400} x^2 = 400x^2.$$

5a. Compute

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4.$$

Notice in particular that we don't have to be able to evaluate the function at a point to compute its limit at that point.

5b. Compute

$$\lim_{x \rightarrow 3^-} \frac{x}{x^2 - 2x - 3} = \lim_{x \rightarrow 3^-} \frac{x}{(x - 3)(x + 1)} = -\infty.$$

5c. We make the substitution $y = 7x$, and our limit becomes

$$\lim_{y \rightarrow 0} \frac{\sin y}{(y/7)} = 7 \lim_{y \rightarrow 0} \frac{\sin y}{y} = 7.$$

You won't lose points on a problem like this if you omit the explicit substitution.

5d. In this case, we rationalize the numerator,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 1} - \sqrt{x + 1}}{x - 1} \cdot \frac{\sqrt{x^2 + 1} + \sqrt{x + 1}}{\sqrt{x^2 + 1} + \sqrt{x + 1}} \\ &= \lim_{x \rightarrow 1} \frac{x^2 + 1 - (x + 1)}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} \\ &= \lim_{x \rightarrow 1} \frac{x^2 - x}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{(x - 1)(\sqrt{x^2 + 1} + \sqrt{x + 1})} \\ &= \lim_{x \rightarrow 1} \frac{x}{(\sqrt{x^2 + 1} + \sqrt{x + 1})} = \frac{1}{2\sqrt{2}}. \end{aligned}$$

5e. According to our rule from class, the following calculation is entirely fair:

$$\lim_{x \rightarrow -\infty} \frac{x^3 - x^2 + 1}{1 - x^2} = \lim_{x \rightarrow -\infty} \frac{x^3}{-x^2} = \lim_{x \rightarrow -\infty} (-x) = +\infty.$$

5f. Since $\sin x$ does not have a limit as $x \rightarrow \infty$ we use the Squeeze Theorem (a.k.a. the Sandwich Theorem), observing

$$-e^{-x} \leq e^{-x} \sin x \leq e^{-x}.$$

We have $\lim_{x \rightarrow \infty} (-e^{-x}) = \lim_{x \rightarrow \infty} (e^{-x}) = 0$, so by the Squeeze Theorem

$$\lim_{x \rightarrow \infty} e^{-x} \sin x = 0.$$

6. First, observe that $\ln(1 - x)$ is only defined for $x < 1$ and $\ln(1 + x)$ is only defined for $x > -1$, so our range is restricted to this interval. Also, we cannot divide by 0, so we must have $x \neq 0$. We conclude that the points of continuity are

$$(-1, 0) \cup (0, 1).$$

7. We observe that the only point at which f may not be continuous is $x = 1$, and at this point $f(1) = 2$. In order to make the function continuous at this point, we must ensure

$$\lim_{x \rightarrow 1^+} x - c = 2,$$

and this requires $c = -1$.

8. We begin by defining the function

$$f(x) = e^x - 2 - \sin x,$$

and we note that our goal will be to show that $f(x)$ has at least one real root. First, we observe that $f(0) = -1$. Next, we observe that since $e > 2$ we know that $e^2 > 4$, so that $f(2) = e^2 - 2 - \sin 2 > 0$. We can conclude from the Intermediate Value Theorem that there is a root on the interval $(0, 2)$.

9. We begin by defining the function

$$f(x) = x^4 + x^3 + x - 1,$$

and we observe that $f(0) = -1$ and $f(1) = 2$, so that we are guaranteed a root in $(0, 1)$. We take

$$c_1 = \frac{1}{2} \pm \frac{1}{2},$$

and compute

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 + \frac{1}{2} - 1 = \frac{1}{16} + \frac{1}{8} + \frac{1}{2} - 1 = \frac{1 + 2 + 8 - 16}{16} = -\frac{5}{16} < 0.$$

We conclude that the root is on the interval $(\frac{1}{2}, 1)$, and our second approximation becomes

$$c_2 = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4} \pm \frac{1}{4},$$

and $\frac{1}{4} < \frac{1}{3}$, so this is a sufficient approximation. (**Note.** This equation has a second real root between -2 and -1, so it's possible to approximate that one instead, which is fine.)

10. Begin by noticing that $x = \sqrt[4]{5}$ is a root of $f(x) = x^4 - 5$. We observe that $f(1) = -4$ and $f(2) = 11$, so that we are guaranteed a root in $(1, 2)$. We take

$$c_2 = \frac{1 + 2}{2} = \frac{3}{2} \pm \frac{1}{2}.$$

This error is not small enough, so we proceed with another step, noting this time $f(\frac{3}{2}) = \frac{81}{16} - 5 = \frac{81-80}{16} = \frac{1}{16}$. Since this is positive we are guaranteed a root in $(1, \frac{3}{2})$. We take

$$c_2 = \frac{1 + \frac{3}{2}}{2} = \frac{5}{4} \pm \frac{1}{4}.$$

Since $\frac{1}{4} < \frac{1}{3}$ $c_2 = \frac{5}{4}$ is sufficient.

11. According to the definition,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cos \frac{1}{h}}{h} = \lim_{h \rightarrow 0} h \cos \frac{1}{h}.$$

Observing now that $|\cos \frac{1}{h}| \leq 1$, we have

$$-|h| \leq h \cos \frac{1}{h} \leq |h|,$$

and so by the squeeze theorem $f'(0) = 0$. (This is a case in which $f(x)$ is differentiable at a point, but $f'(x)$ is not continuous at that point.)

12a. We can see that $f(x)$ is not continuous at $x = 0$ by computing

$$\lim_{x \rightarrow 0^-} f(x) = 1,$$

and

$$\lim_{x \rightarrow 0^+} f(x) = -1,$$

from which we conclude that the limit of $f(x)$ as $x \rightarrow 0$ does not exist. We know that if a function is not continuous at a point then it cannot be differentiable at that point, so $f(x)$ is not differentiable at $x = 0$.

12b. **Method 1.** According to the definition of derivative,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}.$$

Since f is defined by different functions for $h < 0$ and for $h > 0$, we must compute and compare right and left limits:

$$\lim_{h \rightarrow 0^-} \frac{(h^2 + 1) - 1}{h} = \lim_{h \rightarrow 0^-} h = 0,$$

while

$$\lim_{h \rightarrow 0^+} \frac{(2h + 1) - 1}{h} = \lim_{h \rightarrow 0^+} 2 = 2.$$

Since these limits do not agree, we can conclude that $f(x)$ is not differentiable at $x = 0$.

Method 2. In cases for which

$$f(x) = \begin{cases} f_1(x) & x \leq a \\ f_2(x) & x > a \end{cases},$$

where $f_1(a) = f_2(a)$, $f'_1(a) = c_1$, and $f'_2(a) = c_2$ we can proceed as follows: if $c_1 \neq c_2$ then f is not differentiable at $x = a$, while if $c_1 = c_2$ then f is differentiable at $x = a$ and $f'(a) = c_1$. Here, $f_1(x) = x^2 + 1$ and $f_2(x) = 2x + 1$, so $f'(0) = 0$ while $f'_2(0) = 2$. We can draw the same conclusion as we did with Method 1. (Be sure to check all assumptions when using this method; try it, for example, on (2a).)

12c. In this case,

$$f'(0) = \lim_{h \rightarrow 0} \frac{h|h|}{h} = \lim_{h \rightarrow 0} |h| = 0,$$

and so f is differentiable at $x = 0$ with $f'(0) = 0$.

13. The slope of the tangent line is given by the derivative $y'(1) = 3(1)^2 = 3$. We have, then $y - 2 = 3(x - 1)$. See Figure 4.

14a. The graphs are sketched below.

14b. The average velocity is

$$v_{avg} = \frac{s(2) - s(0)}{2 - 0} = \frac{80 - 0}{2} = 40 \frac{\text{km}}{\text{hr}}.$$

14c. The instantaneous velocity is

$$v_{inst} = s'(1) = 40 \frac{\text{km}}{\text{hr}}.$$

15. The slope of the tangent line to this curve at any point x is

$$f'(x) = 2x + 1,$$

and we must find the value x for which this slope is 3 (the slope of the given line). We solve

$$2x + 1 = 3$$

to find $x = 1$. The point is $(1, 3)$.

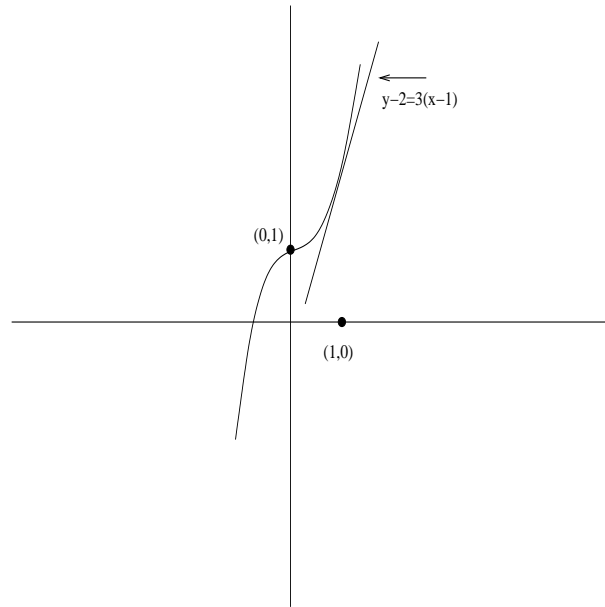


Figure 4: Figure for Problem 13.

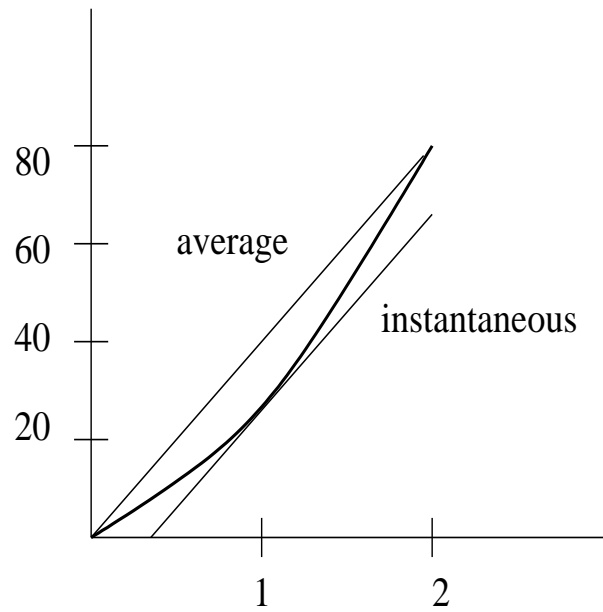


Figure 5: Figure for Problem 14.