Exam 1 Practice Problems

Graphing and scientific calculators will not be allowed on the exam, but you will be allowed to use a four-function calculator for arithmetic computations. The exam will include some multiple choice and some work-out problems. This review should not be used as your sole source for preparation for the exam. You should also rework all examples given in lecture, all homework problems, all recitation assignment problems, and all quiz problems.

1. Evaluate $\displaystyle\int (\ln x)^2\,dx$.

2. Evaluate $\displaystyle\int_0^2 \frac{2x}{(x^2-1)^{1/3}}\,dx$.

3. Evaluate $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$.

4. Evaluate $\displaystyle\int \frac{5x^2+4x+3}{x(x^2+x+1)}\,dx$.

5. Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1} x\,dx$.

6. Evaluate $\displaystyle\int x^5 e^{x^2}\,dx$.

7a. Find the third order Taylor polynomial near $x = 0$ for $f(x) = \tan^{-1} x$.

7b. Use your result from (7a) to approximate the values of $\frac{\pi}{4}$ and then π.

8. Suppose $f(x) = \ln x$ is approximated by a second order Taylor polynomial near $x = 1$. Find the maximum possible error of this approximation for $x \in [1, 2]$.

9. Determine the smallest order n that will ensure that the n^{th} order Taylor polynomial for $f(x) = x \ln x - x$ near $x = 1$ has a maximum error less than 0.01 for $x \in [1, 2]$.

10. Solve the differential equation $\dfrac{dy}{dx} = y \cos x$ with initial condition $y(0) = 1$.

11. Assume that the size of a population evolves according to the logistic equation with intrinsic rate of growth $r = 1.25$ and carrying capacity $K = 500$. (a) Find the differential equation that describes the rate of growth of this population. (b) Find all equilibria, and discuss the stability of the equilibria using the graphical approach. (c) Find the eigenvalues associated with the equilibria and then use the stability criterion to determine stability of the equilibria.

12. For the single-species population model

$$\frac{dp}{dt} = \frac{r}{2}p\left(1 - \left(\frac{p}{K}\right)^2\right); \quad p(0) = p_0,$$

find all equilibrium points (including negative ones) and determine whether or not each is stable using the graphical approach. Also, if $p_0 = \frac{K}{2}$, compute

$$\lim_{t\to\infty} p(t).$$

13. For the differential equation

$$\frac{dy}{dt} = y^2 - 1; \quad y(0) = y_0,$$

identify the stable equilibrium point (there is exactly one) and compute the return time to this point from $y_0 = 0$.

Solutions

1. This can either be done with a substitution ($u = \ln x$) followed by two applications of integration by parts, or we can integrate by parts to begin with. For this latter approach, we set $u = (\ln x)^2$ and $dv = dx$, from which we have $du = 2(\ln x)\frac{1}{x}dx$ and $v = x$. The integral becomes

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2x\ln x + 2x + C,$$

where if you don't remember the antiderivative of $\ln x$ (we did this in class, and we see it often enough that it is worth memorizing), you can compute it by integration by parts ($\int \ln x dx = x\ln x - x + C$).

2. This is an improper integral, with the integrand infinite at $x = 1$. We write

$$\begin{aligned}
\int_0^2 \frac{2x}{(x^2-1)^{1/3}}dx &= \int_0^1 \frac{2x}{(x^2-1)^{1/3}}dx + \int_1^2 \frac{2x}{(x^2-1)^{1/3}}dx \\
&= \lim_{c\to 1^-}\int_0^c \frac{2x}{(x^2-1)^{1/3}}dx + \lim_{c\to 1^+}\int_c^2 \frac{2x}{(x^2-1)^{1/3}}dx \\
&= \lim_{c\to 1^-}\left[\frac{3}{2}(x^2-1)^{2/3}\Big|_0^c\right] + \lim_{c\to 1^+}\left[\frac{3}{2}(x^2-1)^{2/3}\Big|_c^2\right] \\
&= \lim_{c\to 1^-}\left[\frac{3}{2}(c^2-1)^{2/3} - \frac{3}{2}\right] + \lim_{c\to 1^+}\left[\frac{3}{2}3^{2/3} - \frac{3}{2}(c^2-1)\right] \\
&= -\frac{3}{2} + \frac{3^{5/3}}{2}.
\end{aligned}$$

3. Recognizing that the integrand in this case is a rational function, we proceed with partial fractions. The denominator factors as $x^3 + 2x^2 - 3x = x(x+3)(x-1)$, and so we have

$$\frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} = \frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}.$$

Multiplying by the denominator, we find

$$4x^2 + 13x - 9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3).$$

This is a case in which it's easiest to proceed by choosing particular values of x. We use:

$$\begin{aligned} \text{If } x = 0: & \quad -9 = -3A \quad \Rightarrow A = 3 \\ \text{If } x = 1: & \quad 8 = 4C \quad \Rightarrow C = 2 \\ \text{If } x = -3: & \quad -12 = 12B \quad \Rightarrow B = -1. \end{aligned}$$

Finally,

$$\begin{aligned} \int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx &= \int \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1} dx \\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C. \end{aligned}$$

4. Again, we have a problem that should clearly be solved by partial fractions, though in this case the denominator has an irreducible quadratic expression $x^2 + x + 1$. (One way to check that it's irreducible is to use the quadratic formula to see that the equation $x^2 + x + 1 = 0$ has no real roots. In this case, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.) The partial fraction decomposition is

$$\frac{5x^2 + 4x + 3}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}.$$

Multiplying across by the original denominator, we have

$$5x^2 + 4x + 3 = A(x^2 + x + 1) + (Bx + C)x = (A + B)x^2 + (A + C)x + A,$$

for which we apply the "equating coefficients" method:

$$\begin{aligned} \text{constants}: & \quad 3 = A \\ x: & \quad 4 = A + C \\ x^2: & \quad 5 = A + B. \end{aligned}$$

We conclude $A = 3$, $B = 2$, and $C = 1$, so that

$$\frac{5x^2 + 4x + 3}{x(x^2 + x + 1)} = \frac{3}{x} + \frac{2x + 1}{x^2 + x + 1}.$$

This integrates like only an exam problem can:

$$\int \frac{3}{x} dx + \int \frac{2x + 1}{x^2 + x + 1} dx = 3\ln|x| + \ln|x^2 + x + 1| + C.$$

(NOTE: The second integral was computed using a u-substitution.)

5. Most likely you do not know the antiderivative of $\tan^{-1} x$, but you should know its derivative. This leads us to use integration by parts. Take $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{1}{1+x^2}dx$ and $v = x$ and we have

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1} x \; dx &= x \tan^{-1} x \Big|_0^{\frac{1}{\sqrt{3}}} - \int_0^{\frac{1}{\sqrt{3}}} \frac{x}{1+x^2} dx \\
&= \left(\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) - 0 \right) - \frac{1}{2} \int_1^{\frac{4}{3}} \frac{1}{m} dm \\
&= \frac{1}{\sqrt{3}} \left(\frac{\pi}{6} \right) - \frac{1}{2} \ln \mid m \mid \Big|_1^{\frac{4}{3}} \\
&= \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \left(\ln \frac{4}{3} - \ln 1 \right) \\
&= \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln \frac{4}{3}
\end{aligned}$$

(NOTE: The integral obtained using the integration by parts formula was computing by making the substitution $m = 1 + x^2$ and changing the limits of integration: When $x = 0$, $m = 1 + 0^2 = 1$, and when $x = \frac{1}{\sqrt{3}}$, $m = 1 + \left(\frac{1}{\sqrt{3}} \right)^2 = \frac{4}{3}$.)

6. To compute this integral, we must first make an appropriate substitution and then use integration by parts. Let $m = x^2$. Then $dm = 2x \; dx$, which if we solve for dx, we get $dx = \frac{1}{2x} \; dm$. Applying this substitution, we obtain

$$\begin{aligned}
\int x^5 e^{x^2} \; dx &= \int x^5 e^m \left(\frac{1}{2x} \right) \; dm \\
&= \frac{1}{2} \int x^4 e^m \; dm \\
&= \frac{1}{2} \int (x^2)^2 e^m \; dm \\
&= \frac{1}{2} \int m^2 e^m \; dm \\
&= \frac{1}{2} \left(m^2 e^m - 2me^m + 2e^m \right) + C (\text{using integration by parts}) \\
&= \frac{1}{2} x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C
\end{aligned}$$

NOTE: You will need to show your work for the integration by parts step on the exam.

7a. For the third order polynomial, we have

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3,$$

for which we compute:

$$\begin{aligned}
f(x) &= \tan^{-1} x \Rightarrow f(0) = 0 \\
f'(x) &= \frac{1}{x^2+1} \Rightarrow f'(0) = 1 \\
f''(x) &= -\frac{2x}{(x^2+1)^2} \Rightarrow f''(0) = 0 \\
f'''(x) &= -\frac{(x^2+1)^2 2 - 8x^2(x^2+1)}{(x^2+1)^4} \Rightarrow f'''(0) = -2.
\end{aligned}$$

Upon substitution of these values into $P_3(x)$, we conclude

$$P_3(x) = x - \frac{1}{3}x^3.$$

7b. Notice that the conclusion of (7a) is that

$$\tan^{-1} x \approx x - \frac{1}{3}x^3.$$

If we recall that $\tan\frac{\pi}{4} = 1$, we see that $\tan^{-1} 1 = \frac{\pi}{4}$, and so

$$\frac{\pi}{4} \approx 1 - \frac{1}{3}(1)^3 = \frac{2}{3} \Rightarrow \pi \approx \frac{8}{3}.$$

(The approximation can be improved by taking higher order polynomials.)

8. Taylor's Inequality (as discussed in class) with $a = 1$ and interval $1 \leq x \leq 2$ is given by $\mid R_2(x) \mid \leq \dfrac{M}{3!} \mid x - 1 \mid^3$ where $M =$ maximum of $\mid f'''(x) \mid$ on $1 \leq x \leq 2$. Here, $f'''(x) = \frac{2}{x^3}$ which is a decreasing function on $[1, 2]$, so $\mid f'''(x) \mid = \mid \frac{2}{x^3} \mid \leq \mid \frac{2}{1^3} \mid = 2$ for all $x \in [1, 2]$. Also, on this interval $(x-1)^3$ is an increasing function, so $\mid x-1 \mid^3 \leq \mid 2-1 \mid^3 = 1$, so we have $\mid R_2(x) \mid \leq \frac{2}{3!} \cdot 1 = \frac{1}{3}$. Thus, the maximum error is $\frac{1}{3}$.

9. In this problem, we are given a maximum error and we are being asked to find the value of n that produces an error that satisfies this upper limit. We again use Taylor's Inequality with $a = 1$ and interval $[1, 2]$ (as discussed in class)

$$| R_n | \leq \frac{M}{(n+1)!} | x - 1 |^{n+1} \quad \text{where } M = \text{ maximum of } \; | f^{(n+1)}(x) | \; \text{ on } [1, 2]$$

but this time we must find a general form for $f^{(n+1)}(x)$. We start by finding the general form for the n^{th} derivative:

$$\begin{aligned}
f'(x) &= \ln x \\
f''(x) &= \frac{1}{x} \\
f'''(x) &= -\frac{1}{x^2} \\
f^{(4)}(x) &= \frac{2}{x^3} \\
f^{(5)}(x) &= -\frac{3!}{x^4} \\
f^{(n)}(x) &= (-1)^n \frac{(n-2)!}{x^{(n-1)}}.
\end{aligned}$$

This means that $f^{(n+1)}(x) = (-1)^{n+1} \dfrac{(n-1)!}{x^n}$. Then

$$\begin{aligned}
M &= \text{maximum of } \; | f^{(n+1)}(x) | \; \text{ on } [1, 2] \\
&= \text{maximum of } \left| \frac{(n-1)!}{x^n} \right| \text{ on } [1, 2] \quad \text{(absolute values kill the } (-1)^{n+1} \text{ term)} \\
&= (n-1)! \; \left(\text{since } \frac{1}{x^n} \text{ is decreasing for positive } n\right)
\end{aligned}$$

So we have $M = (n-1)!$. Since $y = x - 1$ is increasing on $[1, 2]$, its maximum is attained at $x = 2$, and we now have

$$\begin{aligned}
| R_n(x) | &\leq \frac{M}{(n+1)!} | x - 1 |^{n+1} \\
&= \frac{(n-1)!}{(n+1)!} | x - 1 |^{n+1} \\
&\leq \frac{1}{(n+1)(n)} | 2 - 1 |^{n+1} \\
&= \frac{1}{(n+1)(n)}
\end{aligned}$$

We want $| R_n(x) | \leq 0.01 = \frac{1}{100}$, so we are trying to find n so that

$$\frac{1}{(n+1)(n)} \leq \frac{1}{100}$$

Since $9 \cdot 10 = 90$ and $10 \cdot 11 = 110$, we see that $n = 10$ is the minimum value that works.

10. Separating variables and integrating, we have

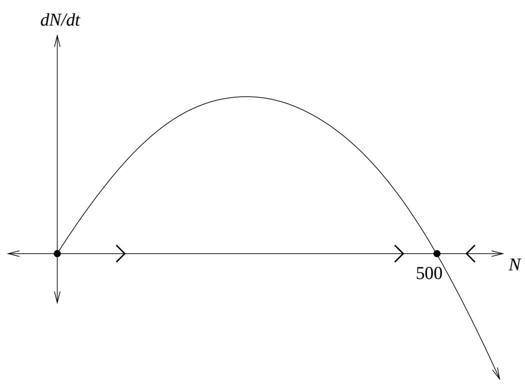
$$\int \frac{1}{y}\,dy = \int \cos x\,dx \Rightarrow \ln|y| = \sin x + C_1 \Rightarrow |y(x)| = e^{\sin x + C_1}.$$

We conclude that $y(x) = Ce^{\sin x}$, where $C = \pm e^{C_1}$ is a constant. Using $y(0) = 1$, we find $y(x) = e^{\sin x}$.

11.(a) The general differential equation for logistic growth is given by $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$ with $N(0) = N_0$, so in this problem we have

$$\frac{dN}{dt} = 1.25N\left(1 - \frac{N}{500}\right) \quad \text{with } N(0) = N_0$$

(b) We set the right-hand side of the differential equation above equal to 0 and solve for N to find equilibria $N = 0$ and $N = 500$. The graph of the function $g(N) = 1.25N\left(1 - \frac{N}{500}\right)$ is a parabola opening downward (because of negative in front of N^2 term) that crosses the horizontal axis at $N = 0$ and $N = 500$. Thus, the graph of $g(N)$ is approximately



NOTE: Since N represents a population, we only draw the graph for $N \geq 0$. For values of N between 0 and 500, $\frac{dN}{dt} > 0$ (since the graph of $g(N) = \frac{dN}{dt}$ lies above the horizontal axis), so the population size N increases on $(0, 500)$. This has been indicated by arrows pointing in the positive direction on the N-axis on $(0, 500)$. For values of N larger than 500, $\frac{dN}{dt} < 0$, so the population size decreases for $N > 500$. This has been indicated by arrows pointing in the negative direction on the N-axis to the right of 500. Using this graphical approach, we see that $N = 0$ is an unstable equilibrium and $N = 500$ is a locally stable equilibrium.

(c) To find the eigenvalues associated with each equilibrium, we first find the derivative of $g(N) = 1.25N\left(1 - \frac{N}{500}\right)$:

$$g(N) = \frac{5}{4}N\left(1 - \frac{N}{500}\right) = \frac{5}{4}N - \frac{1}{400}N^2 \quad \text{so} \quad g'(N) = \frac{5}{4} - \frac{1}{200}N$$

(continued to the next page)

The eigenvalues λ_1 and λ_2 associated with the equilibria $N = 0$ and $N = 500$, respectively, are

$$\lambda_1 = g'(0) = \frac{5}{4} \quad \text{and} \quad \lambda_2 = g'(500) = \frac{5}{4} - \frac{1}{200}(500) = -\frac{5}{4}$$

According to the stability criterion, since $\lambda_1 > 0$, $N = 0$ is unstable, and since $\lambda_2 < 0$, $N = 500$ is locally stable. These results agree with the results obtained from the graphical approach in part (b).

12. We find the equilibrium points by solving

$$\frac{r}{2}p_e\left(1 - \left(\frac{p_e}{K}\right)^2\right) = 0,$$

which has three solutions $p_e = -K, 0, K$. By sketching the graph of the cubic function $g(p) = \frac{r}{2}p\left(1 - \left(\frac{p}{K}\right)^2\right)$, we see that $-K$ and K are both stable and 0 is unstable. We also see that if $p_0 = \frac{K}{2}$, $\lim_{t\to\infty} p(t) = K$. NOTE: On the exam, show your sketch and a brief justification as to how you obtained it. Also, be sure to draw the appropriate directional arrows on the horizontal axis.

13. First, the equilibrium points are $y_e = -1, 1$, and it's clear either from the graphical approach or from the eigenvalue method that only $y_e = -1$ is stable. (You must show this work on the exam.) Recalling that the return time T_R is defined by

$$y(T_R) - y_e = \frac{1}{e}(y_0 - y_e),$$

we have $y(T_R) = -1 + \frac{1}{e}$. Next, in order to get information about time, we invert the differential equation to

$$\frac{dt}{dy} = \frac{1}{y^2 - 1} \Rightarrow \int_0^{T_R} dt = \int_0^{-1+\frac{1}{e}} \frac{dy}{y^2 - 1}.$$

For the integral in y, we use partial fractions to show

$$\frac{1}{y^2 - 1} = \frac{\frac{1}{2}}{y - 1} - \frac{\frac{1}{2}}{y + 1},$$

so that

$$\begin{aligned}\int_0^{-1+\frac{1}{e}} \frac{dy}{y^2 - 1} &= \int_0^{-1+\frac{1}{e}} \frac{\frac{1}{2}}{y - 1} - \frac{\frac{1}{2}}{y + 1} dy \\ &= \frac{1}{2}\ln|y - 1| - \frac{1}{2}\ln|y + 1|\Big|_0^{-1+\frac{1}{e}} \\ &= \frac{1}{2}\ln\Big| - 2 + \frac{1}{e}\Big| - \frac{1}{2}\ln\Big|\frac{1}{e}\Big| \\ &= \frac{1}{2}\ln(2e - 1).\end{aligned}$$