

M147 Practice Problems for Exam 2

Exam 2 will cover sections 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 5.1, and 5.2. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit.

1. Compute the derivative of each of the following functions:

1a.

$$f(x) = x^{\frac{2}{3}} + x^{-7}.$$

1b.

$$f(x) = x \sin x.$$

1c.

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

1d.

$$f(x) = \left(2x + \frac{1}{x}\right)^2.$$

2. Suppose

$$h(x) = f(x)e^{g(x)},$$

and $f(2) = 4$, $f'(2) = 7$, $g(2) = 0$, and $g'(2) = 3$. Compute $h'(2)$.

3. Compute $f''(x)$ if

$$f(x) = \sin(\sqrt{2x}).$$

4. Compute $\frac{dy}{dx}$ given that

$$\sin(xy) = x.$$

Find an equation for the line that is tangent to this curve at the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$.

5. Find $\frac{d^2y}{dx^2}$ if

$$xy - e^y = 0.$$

6. An airplane is flying 6 miles above the ground on a flight path that will take it directly over a radar tracking station. If the distance between the plane and tracking station is decreasing at a rate of 400 miles per hour when the distance is 10 miles, what is the speed of the plane?

7. Suppose that two sides of a triangle are 4 cm and 5 cm in length and that the angle between them is increasing at a rate of .06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides is $\frac{\pi}{3}$.

8. Let

$$f(x) = \cos x - \sin x, \quad -\frac{\pi}{4} \leq x \leq \frac{3\pi}{4},$$

and compute $\frac{df^{-1}}{dx}(1)$.

9. Evaluate the expression

$$\tan(\sin^{-1}(\frac{2}{3})).$$

10. Show that

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < +1.$$

11. Let

$$y = x^{\tan x}, \quad 0 \leq x < \frac{\pi}{2},$$

and compute $\frac{dy}{dx}$.

12. Use a linear approximation to estimate a value for

$$\ln(.99).$$

13. Consider a right triangle with hypotenuse length l and sidelengths 3 and x . Suppose x is measured as $x = 4 \pm .05$, and use linear approximation to approximate the associated range of error on l .

14. Sketch a graph of the function

$$f(x) = |3 - |x||,$$

on the interval $[-3, 1]$ and determine all local and global extrema on this interval.

15. Suppose that $f(x)$ is continuous on the interval $[2, 5]$ and differentiable on the interval $(2, 5)$. Show that if $1 \leq f'(x) \leq 4$ for all $x \in [2, 5]$, then $3 \leq f(5) - f(2) \leq 12$.

16a. For the function

$$f(x) = \frac{x^2}{1+x}; \quad x \neq -1,$$

find the intervals on which f is increasing and the intervals on which x is decreasing.

16b. For the function defined in (16a) find the intervals on which f is concave up and the intervals on which f is concave down.

17. Suppose that $f(x)$ is twice differentiable in an open interval containing the point $x = c$ and has a local minimum at the same point. Show that the function $g(x) = e^{f(x)}$ has a local minimum at $x = c$.

Solutions

1.

1a. Applying the power rule to each summand, we find

$$\frac{d}{dx}(x^{\frac{2}{3}} + x^{-7}) = \frac{2}{3}x^{-\frac{1}{3}} - 7x^{-8}.$$

1b. Applying the product rule, we find

$$\frac{d}{dx}x \sin x = \sin x + x \cos x.$$

1c. Applying the quotient rule, we find

$$\frac{d}{dx} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2},$$

and though considerable simplification is possible, this form is sufficient for the exam.

1d. Proceeding with the chain rule, we set $u = 2x + \frac{1}{x}$ and compute

$$\frac{d}{dx}u^2 = \frac{d}{du}u^2 \frac{du}{dx} = 2u(2 - \frac{1}{x^2}) = 2(2x + \frac{1}{x})(2 - \frac{1}{x^2}).$$

(This substitution does not need to be made explicitly.)

2. First,

$$h'(x) = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x),$$

and so

$$h'(2) = f'(2)e^{g(2)} + f(2)e^{g(2)}g'(2) = 7e^0 + 4e^0 3 = 7 + 12 = 19.$$

3. **Method 1.** Compute directly

$$f'(x) = \cos(\sqrt{2^x}) \frac{1}{2\sqrt{2^x}} 2^x \ln 2 = \frac{\ln 2}{2} \cos(\sqrt{2^x}) \sqrt{2^x},$$

and

$$\begin{aligned} f''(x) &= \frac{\ln 2}{2} \left(-\sin(\sqrt{2^x}) \frac{\ln 2}{2} 2^x + \cos(\sqrt{2^x}) \frac{\ln 2}{2} \sqrt{2^x} \right) \\ &= \left(\frac{\ln 2}{2} \right)^2 \left(\sqrt{2^x} \cos(\sqrt{2^x}) - 2^x \sin(\sqrt{2^x}) \right). \end{aligned}$$

Method 2. First, observe that $\sqrt{2^x} = (\sqrt{2})^x$, which eliminates the need for a nested chain rule. Now,

$$\frac{d}{dx} \sin((\sqrt{2})^x) = \cos((\sqrt{2})^x) (\sqrt{2})^x \ln \sqrt{2},$$

and

$$\begin{aligned} \frac{d^2}{dx^2} \sin((\sqrt{2})^x) &= -\sin((\sqrt{2})^x) ((\sqrt{2})^x \ln \sqrt{2})^2 + \cos((\sqrt{2})^x) ((\sqrt{2})^x \ln \sqrt{2}) \ln \sqrt{2} \\ &= (\ln \sqrt{2})^2 \left(\cos((\sqrt{2})^x) (\sqrt{2})^x - \sin((\sqrt{2})^x) 2^x \right), \end{aligned}$$

which is equivalent to the expression from Method 1.

4. We compute implicitly

$$\frac{d}{dx} \sin(xy) = \frac{d}{dx} x \Rightarrow \cos(xy) \frac{d}{dx} (xy) = 1 \Rightarrow \cos(xy) \left(y + x \frac{dy}{dx} \right) = 1.$$

Solving for $\frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(xy)} - y}{x}.$$

At the point $(\frac{1}{\sqrt{2}}, \frac{\sqrt{2}\pi}{4})$, we have

$$\frac{dy}{dx} = \frac{\frac{1}{\cos(\frac{\pi}{4})} - \frac{\sqrt{2}\pi}{4}}{\frac{1}{\sqrt{2}}} = 2 - \frac{\pi}{2}.$$

The equation for the tangent line is

$$(y - \frac{\sqrt{2}\pi}{4}) = (2 - \frac{\pi}{2})(x - \frac{1}{\sqrt{2}}).$$

5. We begin by computing the x -derivative of the entire equation,

$$y + x \frac{dy}{dx} - e^y \frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = -\frac{y}{x - e^y}.$$

At this stage we can either compute a second derivative directly from this final expression or take another derivative of our original equation. The former approach is the most direct, and we find

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{(x - e^y)\frac{dy}{dx} - y(1 - e^y\frac{dy}{dx})}{(x - e^y)^2} = -\frac{(x - e^y)(-\frac{y}{x - e^y}) - y + ye^y(-\frac{y}{x - e^y})}{(x - e^y)^2} \\ &= \frac{2y(x - e^y) + y^2e^y}{(x - e^y)^3}. \end{aligned}$$

Note. In case you're curious, the second method would look like this:

$$\frac{dy}{dx} + \frac{dy}{dx} + x \frac{d^2y}{dx^2} - e^y \left(\frac{dy}{dx}\right)^2 - e^y \frac{d^2y}{dx^2} = 0,$$

so that

$$\frac{d^2y}{dx^2} = -\frac{2\frac{dy}{dx} - e^y\left(\frac{dy}{dx}\right)^2}{x - e^y}.$$

Finally, we substitute our expression for $\frac{dy}{dx}$ to find

$$\frac{d^2y}{dx^2} = -\frac{2\left(-\frac{y}{x - e^y}\right) - e^y\left(-\frac{y}{x - e^y}\right)^2}{(x - e^y)} = 2\frac{y}{(x - e^y)^2} + e^y\frac{y^2}{(x - e^y)^3} = \frac{2y(x - e^y) + y^2e^y}{(x - e^y)^3},$$

giving the same result.

6. In this case, we are given that $\frac{dr}{dt} = -400$, where r denotes the distance between the plane and the tracking station. If we let x denote the horizontal distance between the plane and

the tracking station, then what we are looking for is $|\frac{dx}{dt}|$, the plane's speed. In order to find a relation between $\frac{dx}{dt}$ and $\frac{dr}{dt}$, we begin by relating x and r . We have

$$x^2 + 36 = r^2.$$

Upon differentiation of this equation with respect to t , we find

$$2x\frac{dx}{dt} = 2r\frac{dr}{dt}.$$

When $r = 10$, we have $x = \sqrt{100 - 36} = 8$, and therefore

$$2(8)\frac{dx}{dt} = 2(10)(-400) \Rightarrow \frac{dx}{dt} = -\frac{8000}{16} = -500.$$

The negative sign indicates that the plane is moving toward the tracking station, but since the problem asks for *speed*, the correct answer is +500.

7. First, observe that what we know is $\frac{d\theta}{dt}$ and what we would like to know is $\frac{dA}{dt}$, where A is the area of the triangle and θ is the angle between the sides of lengths 4 and 5. In order to get a relationship between A and θ , we recall that the area of a triangle is

$$A = \frac{1}{2}bh,$$

where b is the length of the base of the triangle and h is the height of the triangle. If we draw the triangle with the 5 cm side as the base (i.e., $b = 5$) and let h denote the triangle's height, then we immediately find the relation

$$\sin \theta = \frac{h}{4} \Rightarrow h = 4 \sin \theta.$$

We can now write the area as

$$A = \frac{1}{2}(5)4 \sin \theta = 10 \sin \theta.$$

In order to get a relationship between $\frac{dA}{dt}$ and $\frac{d\theta}{dt}$, we take the derivative of each side of this last expression with respect to t . We find

$$\frac{dA}{dt} = 10 \cos \theta \frac{d\theta}{dt}.$$

At $\theta = \frac{\pi}{3}$ (60°), we have $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and consequently

$$\frac{dA}{dt} = 10\left(\frac{1}{2}\right)(.06) = .3 \text{ cm}^2/s.$$

8. First,

$$f'(x) = -\sin x - \cos x.$$

Also, $f(0) = 1 \Rightarrow f^{-1}(1) = 0$. We have, then,

$$\frac{df^{-1}}{dx}(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-1} = -1.$$

9. For calculations like this it's often convenient to set

$$\theta = \sin^{-1} \frac{2}{3}$$

(using θ because this is an angle), so that

$$\sin \theta = \frac{2}{3}.$$

(See Figure 1.)

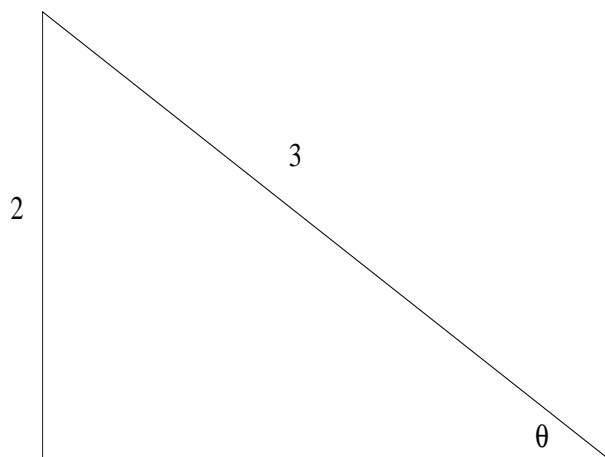


Figure 1: Figure for Problem 9 solution.

According to the Pythagorean Theorem, the adjacent sidelength is

$$b = \sqrt{9 - 4} = \sqrt{5}.$$

In this way,

$$\tan(\sin^{-1} \frac{2}{3}) = \tan \theta = \frac{2}{\sqrt{5}}.$$

10. Set $f(x) = \cos x$ and use the formula

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{-\sin(\cos^{-1} x)}.$$

In order to evaluate $\cos^{-1} x$, set $\theta = \cos^{-1} x$ (we use θ because this is an angle) and note that consequently

$$\cos \theta = x \Rightarrow \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2}.$$

Notice here that since the range of $\cos^{-1} x$ is $[0, \pi]$, we know that $\theta \in [0, \pi]$, and so we know $\sin \theta \geq 0$. This chooses the sign in front of $\sqrt{1 - \cos^2 \theta}$. We finally have

$$\frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1 - x^2}}.$$

11. If we take the natural logarithm of both sides, we have

$$\ln y = \ln x^{\tan x} = (\tan x)(\ln x).$$

Now differentiate each side with respect to x to obtain

$$\frac{1}{y} \frac{dy}{dx} = (\sec^2 x)(\ln x) + \frac{\tan x}{x}.$$

Multiplying this last expression by $y = x^{\tan x}$, we conclude

$$\frac{dy}{dx} = x^{\tan x} \left((\sec^2 x)(\ln x) + \frac{\tan x}{x} \right).$$

12. We start with $f(x) = \ln x$ and use the linear approximation

$$f(x) = f(a) + f'(x)(x - a),$$

where it is reasonable here to take $a = 1$. We find

$$f(x) \approx \ln 1 + 1(x - 1) = x - 1.$$

We can now compute

$$f(.99) \approx .99 - 1 = -.01.$$

(The exact value, to four decimal places, is -.0101.)

13. First, the length l is given by the Pythagorean Theorem,

$$l = \sqrt{3^2 + x^2}.$$

By linear approximation, we have

$$l(x + \Delta x) - l(x) \approx l'(x)\Delta x,$$

where $|l(x + \Delta x) - l(x)|$ is the absolute error, $x = 4$, $\Delta x = .05$ and

$$l'(x) = \frac{x}{\sqrt{x^2 + 9}}.$$

We have, then,

$$l'(4)(.05) = \frac{4}{5}(.05) = .04.$$

We conclude

$$l(4 \pm .05) = 5 \pm .04.$$

14. First, the graph is given in Figure 2. The easiest way to graph a function like this is to expand it out as follows:

$$f(x) = \begin{cases} 3 + x, & -3 \leq x \leq 0 \\ 3 - x, & 0 \leq x \leq 1. \end{cases}$$

Each individual piece is easy to graph. The local minimizers are $x = -3, 1$ and the local minima are $0, 2$. The global minimizer is $x = -3$ and the global minimum is 0 . The local and global maximizer is $x = 0$ and the local and global maximum is 3 .

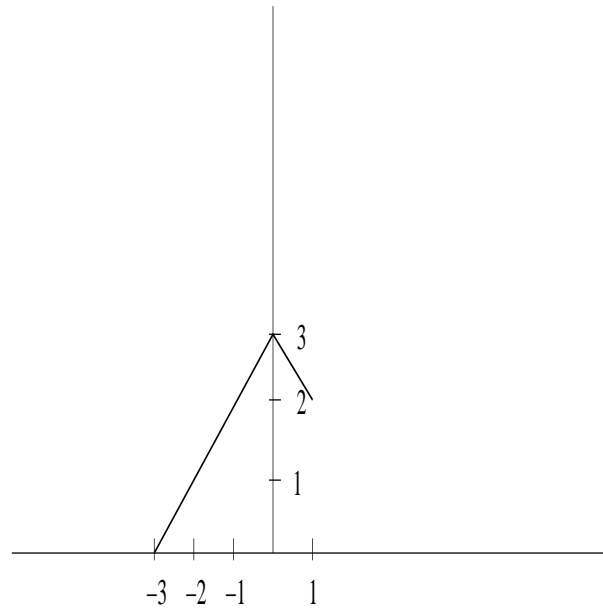


Figure 2: Figure for Problem 14 solution.

15. By the Mean Value Theorem, we know that there exists some value $c \in (2, 5)$ so that

$$f'(c) = \frac{f(5) - f(2)}{3}.$$

Since the largest possible value for $f'(c)$ on this interval is 4 and since the smallest possible value for $f'(c)$ on this interval is 1 , we have the inequality

$$1 \leq \frac{f(5) - f(2)}{3} \leq 4.$$

Multiplying this last inequality by 3 , we find

$$3 \leq f(5) - f(2) \leq 12.$$

16a. First,

$$f'(x) = \frac{(1+x)2x - x^2}{(1+x)^2} = \frac{x^2 + 2x}{(1+x)^2},$$

and from this we can identify the critical points are $x = -2, -1, 0$. Plotting this on a number line, we find that f is decreasing on $[-2, -1) \cup (-1, 0]$ (note that the point where f is undefined is excluded, but the other endpoints are included), and increasing on $(-\infty, -2] \cup [0, +\infty)$.

16b. We compute

$$f''(x) = \frac{(1+x)^2(2x+2) - (x^2+2x)2(1+x)}{(1+x)^4} = \frac{2(x+1)^2 - 2(x^2+2x)}{(1+x)^3} = \frac{2}{(1+x)^3}.$$

The only possible point of inflection is $x = -1$, and plotting this on a number line we find f is concave down on $(-\infty, -1)$ and concave up on $(-1, +\infty)$.

17. By our assumptions on f , we know $f'(c) = 0$ and (by the first derivative test) $f'(x) < 0$ for $x < c$ (and close to c ; i.e., f decreases down to the minimum) and $f'(x) > 0$ for $x > c$ (and close to c). We need to show that precisely the same two conditions hold for $g(x) = e^{f(x)}$. We have

$$g'(x) = e^{f(x)} f'(x) = 0,$$

from which we see that

$$g'(c) = e^{f(c)} f'(c) = 0,$$

and also that $g'(x)$ always has the same sign as $f'(x)$. This means that for x near c we have $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$, and so by the first derivative test g has a minimum at $x = c$.

Note. There is a subtle problem in directly applying the second derivative test because f can have a minimum with $f''(c) = 0$. (E.g., $f(x) = x^4$ with $c = 0$.) In that case we would find $g''(c) = 0$, and this is not enough to conclude that $x = c$ is a minimizer of g .