

M147 Practice Problems for Exam 3

Exam 3 will cover sections 5.3, 5.4, 5.5, 2.1, 2.2, 2.3, 5.6, 6.1, 6.2, and also integration by substitution, which we used in Section 6.2. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit. On the exam you will be given the following identities:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}; \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}; \quad \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

1. Let

$$f(x) = \frac{e^{-x}}{x-1}, \quad x \neq 1.$$

1a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

1b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which it is concave down.

1c. Determine the boundary behavior of f by computing limits as $x \rightarrow \pm\infty$.

1d. Use your information from Parts a-c to sketch a graph of this function.

2. A long rectangular sheet of metal, 12 inches wide, is to be made into a rain gutter by turning up two sides at right angles to the sheet. How many inches should be turned up to give the gutter its greatest capacity?

3. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimized? How should the wire be cut so that the total area is maximized?

4. Compute the following limits.

4a.

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x}.$$

4b.

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 1}).$$

4c.

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1}\right)^x.$$

5. Write down a general expression for the sequence with terms

$$\frac{3}{2}, -\frac{5}{8}, \frac{7}{18}, -\frac{9}{32}, \frac{11}{50}, \dots$$

6. Compute the limit

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}}.$$

7. Find all fixed points for the recursion

$$a_{n+1} = \frac{1}{2}a_n\left(\frac{1}{2} - a_n\right),$$

and use the method of cobwebbing to determine which limit will be achieved from the starting value $a_0 = -1$.

8. Find all fixed points for the recursion

$$x_{t+1} = x_t e^{1-x_t},$$

and determine whether each is unstable or asymptotically stable.

9. The discrete logistic population model is

$$N_{t+1} = N_t + RN_t\left(1 - \frac{N_t}{K}\right).$$

Take $R = 1$ and $K = 10$ and show that one drawback of this model is that it can start with a positive population $N_0 > 0$ and return a negative population N_1 .

10. Use a geometric argument to evaluate the integral

$$\int_{-1}^1 |x| dx.$$

11. Use the method of Riemann sums to evaluate

$$\int_1^4 1 - x^2 dx.$$

12. Express the integral

$$\int_0^2 \sqrt{1 + \sin x} dx$$

as the limit of a Riemann sum. Be sure to define all quantities that appear in your expression.

13. Determine whether or not the Fundamental Theorem of Calculus can be applied to the function

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \end{cases},$$

on the interval $[0, 2]$. If so, find the anti-derivative $F(x)$ and use it to compute

$$\int_0^2 f(x) dx.$$

14. Compute

$$\frac{d}{dx} \int_{\cos x}^{x^2+1} e^{-x^2} dx.$$

15. Evaluate the following indefinite integrals.

15a.

$$\int \cos(2x - 1) dx.$$

15b.

$$\int \frac{x}{x^2 + 1} dx.$$

16. Evaluate the following definite integrals.

16a.

$$\int_0^{\sqrt{\pi}} x \sin(x^2) dx.$$

16b.

$$\int_1^e \frac{\sqrt{\ln x}}{x} dx.$$

16c.

$$\int_{-\frac{\pi}{2}}^{\pi} \sin |x| dx.$$

Solutions.

1a. Compute

$$f'(x) = -\frac{xe^{-x}}{(x-1)^2},$$

and observe that the critical points are $x = 0, 1$. We see that f is increasing on $(-\infty, 0]$ and decreasing on $[0, 1) \cup (1, \infty)$. (We omit $x = 1$ because f isn't defined there.)

1b. Compute

$$f''(x) = \frac{e^{-x}(x^2 + 1)}{(x-1)^3},$$

and observe that the only possible inflection point is $x = 1$. We see that $f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

1c. For the boundary behavior, we have

$$\lim_{x \rightarrow -\infty} \frac{e^{-x}}{x-1} = -\infty$$
$$\lim_{x \rightarrow +\infty} \frac{e^{-x}}{x-1} = 0.$$

1d. In order to anchor the plot, we evaluate f at the critical points, the possible inflection points and the endpoints. We have

$$f(0) = -1$$

$$\lim_{x \rightarrow 1^-} \frac{e^{-x}}{x-1} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{e^{-x}}{x-1} = +\infty.$$

The boundary behavior was obtained in Part c. Here's a sketch:

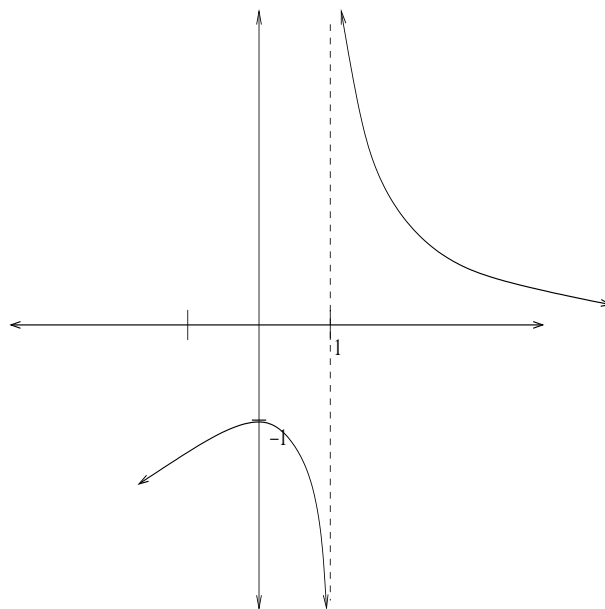


Figure 1: Figure for Problem 1.

2. Let x denote the width of sheet to be turned up on one side and let y denote the width of sheet left flat. If the length of the sheet is L then the volume is

$$V = xyL,$$

where y can be eliminated by the relation $y = 12 - 2x$. (For this problem it seems fairly natural to avoid bringing up the variable y , but I've used it here for consistency with our standard process.) In this way the function we would like to maximize is

$$V(x) = x(12 - 2x)L, \quad 0 \leq x \leq 6.$$

We find the critical points by computing

$$\frac{dV}{dx} = (12 - 4x)L = 0 \Rightarrow x = 3.$$

Evaluating

$$\begin{aligned}V(0) &= 0 \\V(3) &= 18 \\V(6) &= 0,\end{aligned}$$

we conclude that the maximum capacity occurs when $x = 3$ inches are turned up on either side.

3. Let x be the length of each side of the square, and let y be the length of each side of the equilateral triangle. Then the total length of wire is

$$10 = 4x + 3y,$$

while the total area is

$$A = \text{area of square} + \text{area of triangle} = x^2 + \frac{\sqrt{3}}{4}y^2.$$

(You can derive the area formula for an equilateral triangle from the formula $\frac{1}{2}bh$ and either the sidelengths for a 30-60-90 triangle or the Pythagorean theorem. See figure.)

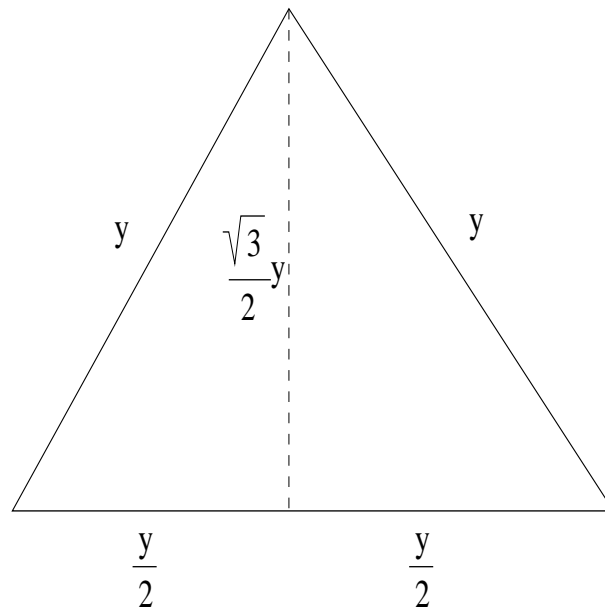


Figure 2: Figure for Problem 3.

Solving our constraint for y , we have

$$y = \frac{10}{3} - \frac{4}{3}x,$$

so that

$$A(x) = x^2 + \frac{\sqrt{3}}{4}\left(\frac{10}{3} - \frac{4}{3}x\right)^2, \quad 0 \leq x \leq \frac{10}{4}.$$

Proceeding as usual, we compute

$$A'(x) = 2x + \frac{\sqrt{3}}{2} \left(\frac{10}{3} - \frac{4}{3}x \right) \left(-\frac{4}{3} \right) \Rightarrow x \left(2 + \frac{8\sqrt{3}}{9} \right) = \frac{20\sqrt{3}}{9}.$$

We conclude that

$$x = \frac{20\sqrt{3}}{18 + 8\sqrt{3}} = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

From our expression for $A'(x)$ we see that $A'(x) < 0$ for $x < \frac{10\sqrt{3}}{9+4\sqrt{3}}$, while $A'(x) > 0$ for $x > \frac{10\sqrt{3}}{9+4\sqrt{3}}$. We conclude that $A(x)$ decreases for all x to the left of this point and increases for all x to the right of it, and is consequently a global minimum. Notice particularly that

$$0 < \frac{10\sqrt{3}}{9 + 4\sqrt{3}} < \frac{10\sqrt{3}}{4\sqrt{3}} = \frac{10}{4},$$

so this value of x is on our domain $0 \leq x \leq \frac{10}{4}$. This says that the area is minimized if the length of wire taken for the square is

$$4x = \frac{40\sqrt{3}}{9 + 4\sqrt{3}}.$$

In order to find the global maximum, we must check $A(x)$ at the two endpoints. We have

$$\begin{aligned} A(0) &= \frac{100\sqrt{3}}{36} \\ A\left(\frac{10}{4}\right) &= \frac{100}{16}. \end{aligned}$$

Notice that

$$\frac{100\sqrt{3}}{36} < \frac{100 \cdot 2}{36} = \frac{100}{18} < \frac{100}{16},$$

so $A\left(\frac{10}{4}\right)$ is larger, and this corresponds with putting all of the wire into the square.

4a.

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = 1.$$

4b.

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 1}) = \lim_{x \rightarrow \infty} x \left(1 - \sqrt{1 - \frac{1}{x^2}} \right) = \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 - \frac{1}{x^2}}}{\frac{1}{x}}.$$

We can now apply l'Hospital's rule to find that this limit is

$$\lim_{x \rightarrow \infty} \frac{-\frac{1}{2} \left(1 - \frac{1}{x^2} \right)^{-\frac{1}{2}} \left(\frac{2}{x^3} \right)}{-\frac{1}{x^2}} = 0.$$

4c.

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1} \right)^x = \lim_{x \rightarrow \infty} e^{\ln \left(\frac{x}{x+1} \right)^x} = \lim_{x \rightarrow \infty} e^{x \ln \left(\frac{x}{x+1} \right)} = e^{\lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+1} \right)}.$$

We compute the limit in the exponent as

$$\lim_{x \rightarrow \infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} -\frac{x^2}{x^2 + x} = -1.$$

We conclude

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x+1}\right)^x = e^{-1}.$$

Note: It's slicker—but for our purposes less instructive—to simply notice that this is the inverse of the limit

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x.$$

5. First, we get the sign right with $(-1)^{n+1}$, $n = 1, 2, \dots$, and we observe that the numerator is $2n + 1$, for $n = 1, 2, \dots$. The easiest way to understand the denominator is to factor out the common factor 2 (a useful trick in general). We find

$$a_n = (-1)^{n+1} \frac{2n+1}{2n^2}, \quad n = 1, 2, \dots$$

6. The important thing to remember here is simply that you compute this sort of limit precisely as with limits in x . More precisely,

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = \lim_{x \rightarrow \infty} x^{\frac{1}{x}}.$$

For the limit in x we can apply l'Hospital's Rule:

$$\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim_{x \rightarrow \infty} \frac{1}{x} \ln x} = e^{\lim_{x \rightarrow \infty} \frac{1}{x}} = e^0 = 1.$$

7. First, the fixed point equation is

$$a = \frac{1}{2}a\left(\frac{1}{2} - a\right) = \frac{1}{4}a - \frac{1}{2}a^2,$$

so that the fixed points are

$$\frac{3}{4}a + \frac{1}{2}a^2 = a\left(\frac{3}{4} + \frac{1}{2}a\right) = 0 \Rightarrow a = 0, -\frac{3}{2}.$$

For the cobwebbing, we can plot $f(a) = \frac{1}{4}a - \frac{1}{2}a^2 = a\left(\frac{1}{4} - \frac{1}{2}a\right)$ by noticing that it's a parabola opening downward with x-intercepts at $a = 0$ and $a = \frac{1}{2}$, and therefore has a maximum value at (the midpoint) $\frac{1}{4}$ of $\frac{1}{4}\left(\frac{1}{4} - \frac{1}{2}\frac{1}{4}\right) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$. We find that for $a_0 = -1$ (see the figure)

$$\lim_{n \rightarrow \infty} a_n = 0.$$

8. First, in order to find the fixed points we solve

$$x = xe^{1-x} \Rightarrow x(1 - e^{1-x}) = 0,$$

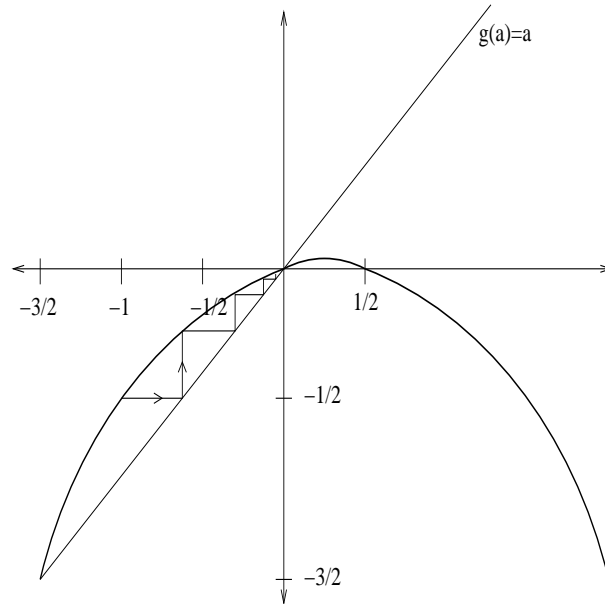


Figure 3: Figure for Problem 7.

from which we have the fixed points

$$x = 0, 1.$$

In order to evaluate the stability of these points, we set

$$f(x) = xe^{1-x},$$

and compute

$$f'(x) = e^{1-x} + xe^{1-x}(-1) = e^{1-x}(1-x).$$

We have:

$$f'(0) = e \Rightarrow |f'(0)| > 1 \Rightarrow x = 0 \text{ is unstable}$$

$$f'(1) = 0 \Rightarrow |f'(0)| < 1 \Rightarrow x = 1 \text{ is asymptotically stable}$$

9. First, for $R = 1$ and $K = 10$ the model becomes

$$N_{t+1} = N_t + N_t\left(1 - \frac{N_t}{10}\right).$$

We see that if N_t is large the second term will be negative, and as a convenient value we can take $N_0 = 50$. We find

$$N_1 = 50 + 50(1 - 5) = 50 - 200 = -150.$$

10. The graph of the function $f(x) = |x|$ looks like a V on $[-1, 1]$, and the area under the curve consists of two triangles with equal areas. Each triangle has baselength 1 and height 1, and so the area of each is $\frac{1}{2}$. We conclude

$$\int_{-1}^1 |x| dx = 1.$$

11. In this case $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$, and we use right endpoints $x_k = 1 + k\Delta x$. We have

$$\begin{aligned} A_n &= \sum_{k=1}^n \left[1 - \left(1 + \frac{3k}{n}\right)^2\right] \frac{3}{n} = \sum_{k=1}^n \left[1 - \left(1 + 6\frac{k}{n} + 9\frac{k^2}{n^2}\right)\right] \frac{3}{n} \\ &= \sum_{k=1}^n -\frac{18k}{n^2} - \frac{27k^2}{n^3} = -\frac{18}{n^2} \sum_{k=1}^n k - \frac{27}{n^3} \sum_{k=1}^n k^2 \\ &= -\frac{18n(n+1)}{n^2 \cdot 2} - \frac{27n(n+1)(2n+1)}{n^3 \cdot 6}. \end{aligned}$$

We conclude

$$\lim_{n \rightarrow \infty} A_n = -9 - 9 = -18.$$

12. The Riemann sum is

$$\int_0^2 \sqrt{1 + \sin x} dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{1 + \sin c_k} \Delta x_k,$$

where P is a partition of the interval $[0, 2]$ with points $P = [x_0, x_1, \dots, x_n]$, $\|P\|$ is the norm of P ($\|P\| = \max_k \Delta x_k$), $\Delta x_k = x_k - x_{k-1}$, and $c_k \in [x_{k-1}, x_k]$ for each $k = 1, \dots, n$.

13. This function is continuous on the interval $[0, 2]$ and so FTC applies. In order to compute the anti-derivative, we first observe that for $x \in [0, 1]$ we have

$$F(x) = \int_0^x y dy = \frac{x^2}{2},$$

as expected. For $x \in [1, 2]$ we must keep in mind that we have

$$F(x) = \int_0^x f(y) dy = \int_0^1 y dy + \int_1^x (2 - y) dy = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \frac{3}{2} = \left(2x - \frac{x^2}{2}\right) - 1.$$

That is,

$$F(x) = \begin{cases} \frac{x^2}{2} & 0 \leq x \leq 1 \\ \left(2x - \frac{x^2}{2}\right) - 1 & 1 < x \leq 2 \end{cases}.$$

Applying FTC, we conclude

$$\int_0^2 f(x) dx = F(2) - F(0) = 1.$$

(This can easily be verified by a geometric argument.)

14. According to Leibniz' rule, we have

$$\frac{d}{dx} \int_{\cos x}^{x^2+1} e^{-x^2} dx = e^{-(x^2+1)^2} 2x - e^{-\cos^2 x} (-\sin x).$$

15a. Use the substitution $u = 2x - 1$, so that $\frac{du}{dx} = 2$. The integral becomes

$$\frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin(2x - 1) + C.$$

15b. Use the substitution $u = x^2 + 1$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int \frac{x \, du}{u \, 2x} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 1| + C,$$

where since $x^2 + 1$ is always positive the absolute value can be dropped.

16a. Use the substitution $u = x^2$, so that $\frac{du}{dx} = 2x$. The integral becomes

$$\int_0^\pi x \sin(u) \frac{du}{2x} = \frac{1}{2} \int_0^\pi \sin u \, du = -\frac{1}{2} \cos u \Big|_0^\pi = 1.$$

16b. Use the substitution $u = \ln x$, so that $\frac{du}{dx} = \frac{1}{x}$. The integral becomes

$$\int_1^e \frac{\sqrt{u}}{x} x \, du = \int_0^1 u^{\frac{1}{2}} \, du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}.$$

16c. We proceed by writing $\sin |x|$ as

$$\sin |x| = \begin{cases} -\sin x, & -\frac{\pi}{2} \leq x \leq 0 \\ \sin x, & 0 < x \leq \pi \end{cases}.$$

In this way, we can compute

$$\int_{-\frac{\pi}{2}}^\pi \sin |x| \, dx = -\int_{-\frac{\pi}{2}}^0 \sin x \, dx + \int_0^\pi \sin x \, dx = \cos x \Big|_{-\frac{\pi}{2}}^0 - \cos x \Big|_0^\pi = 1 - (-1 - 1) = 3.$$