M147 Practice Problems for Final Exam

The final exam for M147 will be comprehensive, covering all sections from the course. Calculators will not be allowed on the exam. The first ten problems on the exam will be multiple choice. Work will not be checked on these problems, so you will need to take care in marking your solutions. For the remaining problems unjustified answers will not receive credit. On the exam you will be given the following identities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}; \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

1. Compute each of the following limits:

1a.

$$\lim_{x\to 2^-} \frac{x}{x^2+3x-10}.$$

1b.

$$\lim_{x\to 0} x e^{\sin(\frac{1}{x})}.$$

1c.

$$\lim_{x\to 0} \frac{x\sin x}{(1-e^x)^2}.$$

1d.

$$\lim_{x\to\infty} [(x+1)^{1/3} - x^{1/3}].$$

1e. The *geometric mean* of two positive real numbers a and b is defined as $\sqrt{ab}$. Show that

$$\sqrt{ab} = \lim_{x\to\infty} (\frac{a^{1/x}+b^{1/x}}{2})^x.$$

2a. Find a value for c that makes the given function continuous at all points.

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}.$$

2b. Determine whether or not your function from (2a) is differentiable at $x = 0$. If it is differentiable at this point, compute its derivative there.

3. Compute the derivative of each of the following functions.

3a.

$$f(x) = \frac{1+\sin^2 x}{x\cos x}.$$

3b.

$$f(x) = \tan^{-1}(2^{\sqrt{x^2+1}}).$$

4. Find an equation for the line that is tangent to the graph of

$$f(x) = \frac{xe^x}{1+x^2}$$

at the point $x = 0$.

5. Suppose the angle of elevation of the Sun is decreasing at a rate of .25 rad/hr. How fast is the shadow cast by a 400 ft tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{6}$?

6. Suppose $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Show that if $f'(x) = x$ for all $x \in (a, b)$, then there exists some value $c \in (0, 1)$ so that

$$f(b) - f(a) = c(b - a).$$

7. Let

$$f(x) = x^{1/3}(x+3)^{2/3}, \quad -\infty < x < \infty.$$

7a. Locate the critical points of f and determine the intervals on which f is increasing and the intervals on which f is decreasing.

7b. Locate the possible inflection points for f and determine the intervals on which f is concave up and the intervals on which it is concave down.

7c. Evaluate f at the critical points and at the possible inflection points, and determine the boundary behavior of f by computing limits as $x \to \pm\infty$.

7d. Use your information from Parts a-c to sketch a graph of this function.

8. Find the side-lengths that maximize the area of an isosceles triangle with given perimeter $P = 10$. (An isosceles triangle is a triangle with two sidelengths equal.)

9. Find all fixed points for the recursion equation

$$a_{n+1} = \frac{3}{4}a_n + \frac{1}{a_n}.$$

Sketch a graph of the function $f(a) = \frac{3}{4}a + \frac{1}{a}$, and use the method of cobwebbing to determine whether or not one of these fixed points will be achieved from the starting value $a_0 = \frac{1}{2}$.

10. Find all fixed points for the recursion equation

$$x_{t+1} = 1 + \frac{2}{x_t}$$

and determine whether or not each is unstable or asymptotically stable.

11. Suppose a function $f(x)$ is continuous on the interval $[0, 1]$ and that you are given the values in Table 1:

Use an appropriate Riemann sum to approximate $\int_0^1 f(x)dx$.

x	$f(x)$
$1/8$	$1/2$
$3/8$	$1/3$
$5/8$	-1
$7/8$	-2

Table 1: Values of $f(x)$ for Problem 11.

12. Use the method of Riemann sums to evaluate

$$\int_1^2 x + x^2 dx.$$

13. Evaluate the following indefinite integrals.

13a.

$$\int e^x \cos(e^x) dx.$$

13b.

$$\int \frac{x}{\sqrt{1+x}} dx$$

13c.

$$\int \cos^{-1} x dx.$$

14. Evaluate the following definite integrals.

14a.

$$\int_1^3 \frac{x^2}{\sqrt{1+x^3}} dx.$$

14b.

$$\int_0^{\frac{\pi}{4}} x \sec^2 x dx.$$

15. Evaluate the following indefinite integral

$$\int \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 x}} dx.$$

16. Find the area of the region bounded by the graphs of $y = x^4$ and $y = 20 - x^2$.

17. Find the total area between the curves $y = x^2$ and $y = 2 - x$ for $x \in [0, 2]$.

18. Find the volume obtained when the region between the graphs of $y = e^x$ and $y = e^{-x}$, $x \in [0, 2]$, is rotated about the x-axis.

19. Find the volume of the solid obtained by rotating about the x-axis the area between the graph of $f(x) = \sqrt{x}$ and the x-axis for $x \in [0, 1]$.

20. Find the volume obtained by rotating the region between $y = 2$ and $y = \sqrt{x}$ for $x \in [0, 4]$ about the x-axis.

21. Find the volume obtained by rotating the region bounded by the curves $y^2 = x$ and $y = \frac{x}{2}$ about the y-axis.

22. Compute the average value of the function

$$f(x) = x + \frac{1}{x}$$

for $x \in [1, 3]$.

23. Determine the length of the graph of $f(x) = x^{\frac{3}{2}} + 1$ for $x \in [0, 4]$.

Solutions

1a. We have

$$\lim_{x \to 2^-} \frac{x}{x^2 + 3x - 10} = \lim_{x \to 2^-} \frac{x}{(x-2)(x+5)} = -\infty,$$

where we have observed that $x - 2$ is negative for x to the left of 2.

1b. We apply the Squeeze Theorem in this case, using the inequality

$$-|x|e \leq xe^{\sin(\frac{1}{x})} \leq |x|e.$$

We have

$$\lim_{x \to 0} -|x|e = \lim_{x \to 0} |x|e = 0,$$

and so according to the Squeeze Theorem

$$\lim_{x \to 0} xe^{\sin(\frac{1}{x})} = 0.$$

1c. We apply L'Hospital's Rule twice,

$$\begin{aligned} \lim_{x \to 0} \frac{x \sin x}{(1 - e^x)^2} &= \lim_{x \to 0} \frac{\sin x + x \cos x}{2(1 - e^x)(-e^x)} = \lim_{x \to 0} \frac{\sin x + x \cos x}{2e^{2x} - 2e^x} \\ &= \lim_{x \to 0} \frac{2 \cos x - x \sin x}{4e^{2x} - 2e^x} = 1. \end{aligned}$$

1d. This limit has the indeterminate form $\infty - \infty$, so the first thing we do is rearrange it into an expression with the form $\frac{0}{0}$. We have

$$\begin{aligned} \lim_{x \to \infty} (x+1)^{1/3} - x^{1/3} &= \lim_{x \to \infty} x^{1/3}[(1 + \frac{1}{x})^{1/3} - 1] \\ &= \lim_{x \to \infty} \frac{(1 + \frac{1}{x})^{1/3} - 1}{x^{-1/3}} = \lim_{x \to \infty} \frac{\frac{1}{3}(1 + \frac{1}{x})^{-2/3}(-\frac{1}{x^2})}{-\frac{1}{3}x^{-4/3}} \\ &= \lim_{x \to \infty} (1 + \frac{1}{x})^{-2/3} \frac{1}{x^{2/3}} = 0. \end{aligned}$$

1e. We observe that this limit has the general form 1^∞, and so we can apply L'Hospital's rule. We have

$$\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}}{2}\right)^x = \lim_{x\to\infty} e^{\ln(\frac{a^{1/x}+b^{1/x}}{2})^x} = \lim_{x\to\infty} e^{x\ln(\frac{a^{1/x}+b^{1/x}}{2})}$$
$$= e^{\lim_{x\to\infty} x\ln(\frac{a^{1/x}+b^{1/x}}{2})}.$$

In order to compute this limit, we write

$$\lim_{x\to\infty} x\ln\left(\frac{a^{1/x}+b^{1/x}}{2}\right) = \lim_{x\to\infty}\frac{\ln(\frac{a^{1/x}+b^{1/x}}{2})}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{2}{a^{1/x}+b^{1/x}}\left(\frac{1}{2}a^{1/x}(\ln a)(-\frac{1}{x^2})+\frac{1}{2}b^{1/x}(\ln b)(-\frac{1}{x^2})\right)}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{1}{a^{1/x}+b^{1/x}}(a^{1/x}\ln a + b^{1/x}\ln b) = \frac{1}{2}(\ln a+\ln b),$$

where in this last step we have used that $\frac{1}{x}\to 0$ as $x\to\infty$. The limit is

$$e^{\frac{1}{2}(\ln a+\ln b)} = e^{\frac{1}{2}\ln(ab)} = e^{\ln(ab)^{1/2}} = \sqrt{ab}.$$

2a. Since

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

we can make this function continuous at all points by choosing $c = 1$.

2b. Since the function is separately defined at $x = 0$, we must proceed from the definition of differentiation. We compute

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{\sin h}{h}-1}{h}$$
$$= \lim_{h\to 0}\frac{\sin h - h}{h^2} = \lim_{h\to 0}\frac{\cos h - 1}{2h} = \lim_{h\to 0}\frac{-\sin h}{2} = 0,$$

where the last two steps both used L'Hospital's rule. We conclude that this function is differentiable at $x = 0$, and that $f'(0) = 0$.

3a. We combine the quotient rule with the product rule to compute

$$f'(x) = \frac{x\cos x(2\sin x\cos x) - (1+\sin^2 x)(\cos x - x\sin x)}{(x\cos x)^2}$$
$$= \frac{2x\sin x\cos^2 x - \cos x + x\sin x - \sin^2 x\cos x + x\sin^3 x}{(x\cos x)^2}.$$

3b. This is a nested chain rule. We have

$$f'(x) = \frac{1}{1+2^{2\sqrt{x^2+1}}}2^{\sqrt{x^2+1}}\frac{x\ln 2}{\sqrt{x^2+1}}.$$

Notice that we can simplify $2^{2\sqrt{x^2+1}}$ as $4^{\sqrt{x^2+1}}$.

4. First,

$$f'(x) = \frac{(1+x^2)(e^x + xe^x) - xe^x(2x)}{(1+x^2)^2} \Rightarrow f'(0) = 1,$$

which is the slope of the tangent line. Using $f(0) = 0$ and the general point-slope form $y - f(a) = f'(a)(x - a)$, we conclude

$$y = x.$$

5. First, observe that what we know is $\frac{d\theta}{dt} = -.25$ rad/hr and what we want to know is $\frac{dx}{dt}$, where x is the length of the shadow (see the diagram in Figure 1).

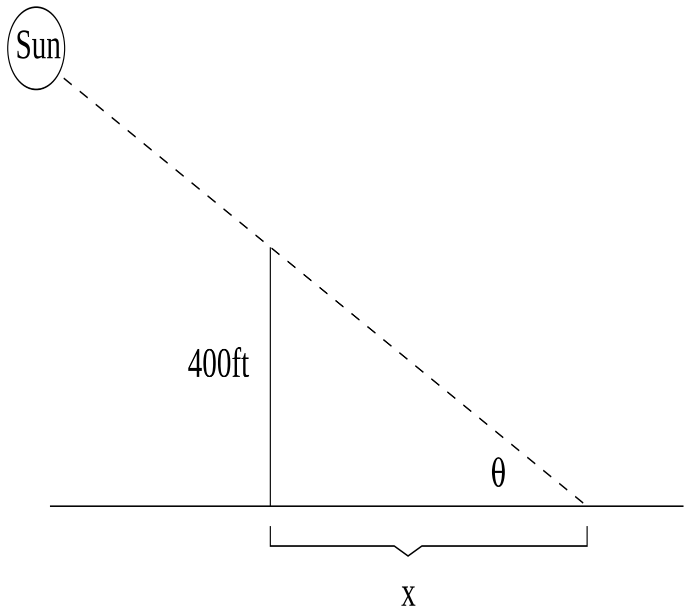

Figure 1: Figure for Problem 5.

We see that the relation between θ and x is

$$\tan\theta = \frac{400}{x}.$$

Upon taking a derivative of this equation with respect to t, we obtain

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt},$$

where we can now fix $\theta = \frac{\pi}{6}$, so that $\sec^2\theta = \frac{1}{\cos^2\frac{\pi}{6}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$, while $x = \frac{400}{\tan\frac{\pi}{6}} = 400\sqrt{3}$. Combining these observations, we have

$$\frac{dx}{dt} = -\frac{x^2}{400}\frac{d\theta}{dt}\sec^2\frac{\pi}{6} = -3(400)(-.25)\frac{4}{3} = +400 \text{ ft/hr}.$$

6. According to the Mean Value Theorem there exists some value $c \in (a, b)$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In this case $f'(c) = c$, and so we conclude

$$\frac{f(b) - f(a)}{b - a} = c \Rightarrow f(b) - f(a) = c(b - a).$$

7a. The derivative of $f(x)$ is

$$f'(x) = \frac{x + 1}{x^{2/3}(x + 3)^{1/3}},$$

from which we find the critical points $x = -3, -1, 0$. We see that f is increasing on $(-\infty, -3] \cup [-1, \infty)$ and decreasing on $[-3, -1]$.

7b. The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{x^{5/3}(x + 3)^{4/3}},$$

from which we find that the possible inflection points are $x = 0, -3$. We see that f is concave up on $(-\infty, -3) \cup (-3, 0)$ and concave down on $(0, \infty)$.

7c. Evaluating f at the critical points, possible inflection points, and at the endpoints, we have:

$$\begin{aligned} f(-3) &= 0 \\ f(-1) &= -2^{2/3} \\ f(0) &= 0 \\ \lim_{x \to -\infty} x^{1/3}(x + 3)^{2/3} &= -\infty \\ \lim_{x \to \infty} x^{1/3}(x + 3)^{2/3} &= +\infty. \end{aligned}$$

7d. Your plot should look something like Figure 2.

8. Let y denote the length of the sides of equal length, and let x denote the length of the side between them. Then the perimeter is

$$10 = 2y + x \Rightarrow y = 5 - \frac{1}{2}x.$$

By the Pythagorean Theorem, the height of such a triangle is $h = \sqrt{y^2 - \frac{1}{4}x^2}$, and so the area to be maximized is

$$A = \frac{1}{2}x\sqrt{y^2 - \frac{1}{4}x^2} \Rightarrow A(x) = \frac{1}{2}x\sqrt{(5 - \frac{1}{2}x)^2 - \frac{1}{4}x^2} = \frac{1}{2}x\sqrt{25 - 5x}, \quad 0 \leq x \leq 5.$$

(The upper limit of 5 is clear both because a value of x larger than this would put a negative number under the radical, and because the single side cannot be more than half the perimeter.) In order to maximize $A(x)$, we compute

$$A'(x) = \frac{\frac{25}{2} - \frac{15}{4}x}{\sqrt{25 - 5x}}.$$

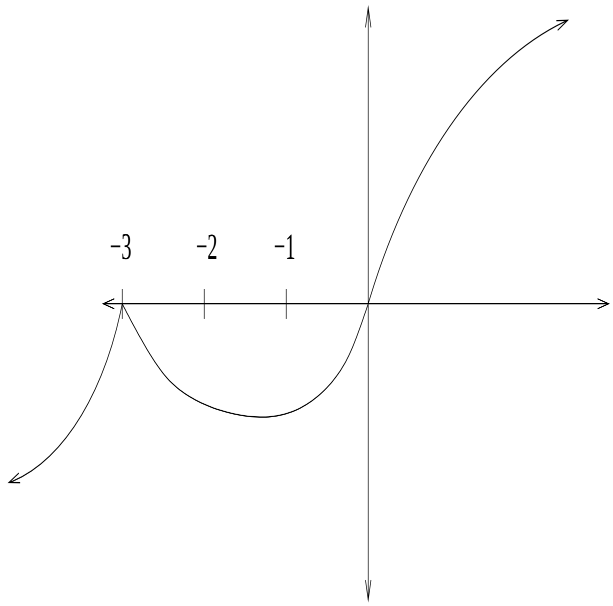

Figure 2: Figure for Problem 7.

The critical values are $x = \frac{10}{3}, 5$, where we observe that $x = 5$ is also a boundary value. Checking $A(x)$ at the critical and boundary values, we find

$$\begin{aligned} A(0) &= 0 \\ A(\frac{10}{3}) &= \frac{5}{3}\sqrt{\frac{25}{3}} = \frac{25}{3\sqrt{3}} \\ A(5) &= 0. \end{aligned}$$

We conclude that the maximum area is $\frac{25}{3\sqrt{3}}$ and the side-lengths are $x = \frac{10}{3}$ and $y = 5 - \frac{1}{2}(\frac{10}{3}) = \frac{10}{3}$. That is, an equilateral triangle.

9. The fixed points solve

$$a = \frac{3}{4}a + \frac{1}{a} \Rightarrow \frac{1}{4}a = \frac{1}{a} \Rightarrow a^2 = 4.$$

We conclude that the fixed points are ± 2. In order to use cobwebbing, we must sketch a graph of the function

$$f(a) = \frac{3}{4}a + \frac{1}{a}.$$

First, setting

$$f'(a) = \frac{3}{4} - \frac{1}{a^2} = 0,$$

we find that the critical points are $a = \pm\frac{2}{\sqrt{3}}, 0$. The function is increasing on $(-\infty, -\frac{2}{\sqrt{3}}] \cup [\frac{2}{\sqrt{3}}, \infty)$ and decreasing on $[-\frac{2}{\sqrt{3}}, 0) \cap (0, \frac{2}{\sqrt{3}}]$. Next,

$$f''(a) = \frac{2}{a^3},$$

and so the only possible point of inflection is $a = 0$. The function is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Finally,

$$\lim_{a \to -\infty} (\frac{3}{4}a + \frac{1}{a}) = -\infty$$
$$f(-\frac{2}{\sqrt{3}}) = -\sqrt{3}$$
$$\lim_{x \to 0^-} (\frac{3}{4}a + \frac{1}{a}) = -\infty$$
$$\lim_{x \to 0^+} (\frac{3}{4}a + \frac{1}{a}) = +\infty$$
$$f(\frac{2}{\sqrt{3}}) = \sqrt{3}$$
$$\lim_{a \to -\infty} (\frac{3}{4}a + \frac{1}{a}) = \infty$$

The plot of this function and the cobwebbing are depicted below in Figure 3. We conclude

$$\lim_{n \to \infty} a_n = 2.$$

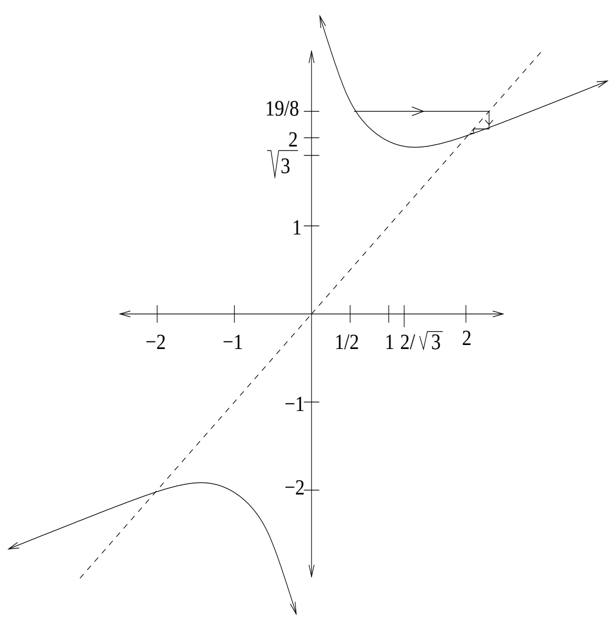

Figure 3: Figure for Problem 9.

10. In order to find the fixed points, we solve

$$x = 1 + \frac{2}{x},$$

which becomes (upon multiplication by x)

$$x^2 - x - 2 = (x - 2)(x + 1) = 0,$$

and the fixed points are $x = -1, 2$. In order to check for stability we set $f(x) = 1 + \frac{2}{x}$, and compute

$$f'(x) = -\frac{2}{x^2}.$$

We have

$$\begin{aligned} f'(-1) &= -2 \Rightarrow -1 \text{ is unstable} \\ f'(2) &= -\frac{1}{2} \Rightarrow 2 \text{ is stable.} \end{aligned}$$

11. Since no value for $f(x)$ is given at either $x = 0$ or at $x = 1$, we cannot take a Riemann sum with left or right endpoints. We see, however, that the values of x are precisely the midpoints of the subintervals in the partition $P = [0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]$. The most reasonable Riemann sum is

$$\sum_{k=1}^{4} f(c_k)\Delta x_k,$$

where the c_k are the interval midpoints. That is,

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = (\frac{1}{2} + \frac{1}{3} - 1 - 2)\frac{1}{4} = -\frac{13}{24}.$$

12. In this case $\triangle x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$, and we use right endpoints $x_k = 1 + k\triangle x$. We have

$$\begin{aligned} A_n &= \sum_{k=1}^{n} [(1 + k\Delta x) + (1 + k\Delta x)^2]\Delta x \\ &= \sum_{k=1}^{n} [(1 + \frac{k}{n}) + (1 + 2\frac{k}{n} + \frac{k^2}{n^2})]\frac{1}{n} \\ &= \Big[\frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{n}\sum_{k=1}^{n} 1 + \frac{2}{n^2}\sum_{k=1}^{n} k + \frac{1}{n^3}\sum_{k=1}^{n} k^2\Big] \\ &= \Big[1 + \frac{1}{n^2}\frac{n(n+1)}{2} + 1 + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\Big]. \end{aligned}$$

Finally,

$$\lim_{n\to\infty} A_n = 1 + \frac{1}{2} + 1 + 1 + \frac{1}{3} = \frac{23}{6}.$$

13a. Using the substitution $u = e^x$, for which $du = e^x dx$, we find

$$\int \cos u du = \sin u + C = \sin(e^x) + C.$$

13b. We make the substitution $u = 1 + x$, with $du = dx$, and obtain

$$\begin{aligned}\int \frac{x}{\sqrt{u}} du = \int \frac{u-1}{\sqrt{u}} du = \int u^{1/2} - u^{-1/2} du \\ = \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + C = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C \\ = (1+x)^{1/2}(\frac{2}{3}x - \frac{4}{3}) + C.\end{aligned}$$

13c. In this case, integrate by parts with $u = \cos^{-1} x$ and $dv = dx$, for which we have $du = -\frac{1}{\sqrt{1-x^2}} dx$ and $v = x$. The integral becomes

$$x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx.$$

For the remaining integral, we use fast substitution (since u has already been used) to obtain

$$x \cos^{-1} x - \sqrt{1-x^2} + C.$$

14a. We make the substitution $u = 1 + x^3$ (or alternatively use fast substitution), so that $du = 3x^2 dx$, and the integral becomes

$$\int_2^{28} \frac{x^2}{\sqrt{u}} \frac{du}{3x^2} = \frac{1}{3} \int_2^{28} u^{-1/2} du = \frac{1}{3} \frac{u^{1/2}}{1/2} \Big|_2^{28} = \frac{2}{3}[\sqrt{28} - \sqrt{2}].$$

14b. We integrate by parts, setting

$$\begin{aligned} u &= x & dv &= \sec^2 x dx \\ du &= dx & v &= \tan x. \end{aligned}$$

We obtain

$$\begin{aligned}\int_0^{\frac{\pi}{4}} x \sec^2 x dx &= x \tan x \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x dx \\ &= \frac{\pi}{4} + \ln|\cos x| \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln(\frac{\sqrt{2}}{2}).\end{aligned}$$

15. We make the substitution $u = 1 + \sin^2 x$, with $du = 2 \sin x \cos x dx$, and we find

$$\int \frac{\sin^3 x \cos x}{\sqrt{u}} \frac{du}{2 \sin x \cos x} = \frac{1}{2} \int \frac{\sin^2 x}{\sqrt{u}} du.$$

At this point we observe that $\sin^2 x = u - 1$, so we have

$$\frac{1}{2} \int \frac{u-1}{\sqrt{u}} du = \frac{1}{2} \int u^{1/2} - u^{-1/2} du = \frac{1}{2}[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}] = \frac{1}{3}(1 + \sin^2 x)^{3/2} - (1 + \sin^2 x)^{1/2} + C.$$

16. First, we locate the points of intersection by solving

$$x^4 = 20 - x^2 \Rightarrow x^4 + x^2 - 20 = 0.$$

In general, fourth order equations are difficult to solve algebraically, but this is really a second order equation in the variable x^2, and it factors as

$$(x^2 - 4)(x^2 + 5) = 0,$$

so that the real roots are $x = \pm 2$. We observe that the upper graph is always $y = 20 - x^2$, and also take advantage of symmetry to compute the area as

$$A = 2\int_0^2 (20 - x^2) - x^4 dx = 2[20x - \frac{x^3}{3} - \frac{x^5}{5}]_0^2 = 2[40 - \frac{8}{3} - \frac{32}{5}] = \frac{928}{15}.$$

17. Plotting these two curves together, we can see that they intersect at $x = 1$, and that for $x < 1$, $y = 2 - x$ is larger, while for $x > 1$, $y = x^2$ is larger. The area between the curves is

$$\begin{aligned} A &= \int_0^1 (2 - x) - x^2 dx + \int_1^2 x^2 - (2 - x) dx \\ &= (2x - \frac{x^2}{2} - \frac{x^3}{3})\Big|_0^1 + (\frac{x^3}{3} - 2x + \frac{x^2}{2})\Big|_1^2 \\ &= 3. \end{aligned}$$

18. We observe that the graph of $y = e^x$ is always above the graph of $y = e^{-x}$ on $[0, 2]$, and so according to the method of washers,

$$\begin{aligned} V &= \pi \int_0^2 (e^x)^2 - (e^{-x})^2 dx = \pi \int_0^2 e^{2x} - e^{-2x} dx \\ &= \frac{\pi}{2}[e^{2x} + e^{-2x}]\Big|_0^2 = \frac{\pi}{2}[e^4 + e^{-4} - 2], \end{aligned}$$

where in this case we used fast substitution.

19. Since the object is being created by rotation, the cross section at each point x is a circle with radius $f(x)$. The area of the cross section at point x is $A(x) = \pi f(x)^2 = \pi x$. Recalling that our volume formula is

$$V = \int_a^b A(x) dx,$$

we compute

$$V = \int_0^1 \pi x dx = \pi \frac{1}{2} x^2 \Big|_0^1 = \frac{\pi}{2}.$$

20. In this case, we use the method of washers, for which we have

$$\begin{aligned} V &= \pi \int_0^4 f(x)^2 - g(x)^2 dx \\ &= \pi \int_0^4 2^2 - (\sqrt{x})^2 dx = \pi \Big[4x - \frac{x^2}{2}\Big]_0^4 \\ &= 8\pi. \end{aligned}$$

21. First, we find the points at which these curves intersect by solving

$$(\frac{x}{2})^2 = x \Rightarrow \frac{x^2}{4} - x = 0 \Rightarrow x(\frac{x}{4} - 1) = 0 \Rightarrow x = 0, 4.$$

The points of intersection are $(0,0)$ and $(4,2)$. If we rotate the region between these curves about the y-axis the line $x = 2y$ describes the outer radius while the parabola $x = y^2$ describes the inner radius. The volume is

$$V = \pi \int_0^2 (2y)^2 - (y^2)^2 dy = \pi \int_0^2 4y^2 - y^4 dy = \pi[\frac{4}{3}y^3 - \frac{1}{5}y^5]_0^2 = \pi[\frac{32}{3} - \frac{32}{5}] = \frac{64\pi}{15}.$$

22. We compute

$$f\text{avg} = \frac{1}{2}\int_1^3 x + x^{-1} dx = \frac{1}{2}[\frac{1}{2}x^2 + \ln|x|]\Big|_1^3 = \frac{1}{2}[\frac{9}{2} + \ln 3] - \frac{1}{2}[\frac{1}{2}] = 2 + \ln\sqrt{3}.$$

23. First, observe that

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}.$$

The formula for arclength is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx,$$

so we have

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x} dx.$$

We carry out this integral with substitution, setting $u = 1 + \frac{9}{4}x$ so that $\frac{du}{dx} = \frac{9}{4}$. The integral becomes

$$L = \int_1^{10} u^{\frac{1}{2}} \frac{4}{9} du = \frac{4}{9} \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^{10} = \frac{8}{27}(10^{\frac{3}{2}} - 1).$$