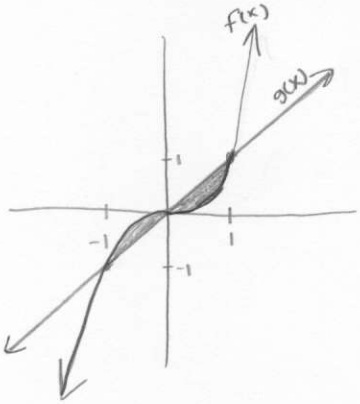
Math 142 - Exam 3 Review

NOTE: Exam 3 covers sections 5.4-5.6, 6.1, 6.2, 6.4, 6.5, 7.1, and 7.2. This review is intended to highlight the material covered on Exam 3 but should not be used as your sole source of practice. Also refer to your instructor's lecture notes, previous week-in-reviews, suggested homework, supplemental homework, and the online homework as additional sources for review and exam preparation.

1. Find the area bounded between $f(x) = x^3$ and $g(x) = x$.



$$\text{Area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$

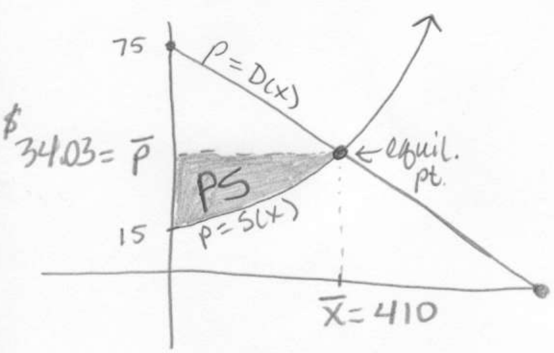
$$= \text{fnInt}(x^3 - x, X, -1, 0) + \text{fnInt}(x - x^3, X, 0, 1)$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \boxed{\frac{1}{2}}$$

2. The price-demand equation for a certain product is given by $p = D(x) = 75 - 0.1x$ dollars per item, and the price-supply equation for this product is $p = S(x) = 15e^{0.002x}$ dollars per item.

   (a) Find the market equilibrium point. Round the equilibrium quantity to the nearest item, and round the equilibrium price to the nearest cent.



$Y_1 = 75 - 0.1x$
$Y_2 = 15e^{0.002x}$ } Graph, Calc. Intersect

$X = 409.65794 \rightarrow 410$ items
$y = 34.034206 \rightarrow \$34.03$ per item

Equilibrium point (rounded) $= \boxed{(410, 34.03)}$

   (b) Using your rounded answers in (a), find the producers' surplus at the equilibrium price level. What does this number represent?

$$PS = \int_0^{410} \left(34.03 - 15e^{0.002x}\right) dx$$

$$= \text{fnInt}(34.03 - 15e^{0.002X}, X, 0, 410)$$

$= \boxed{\$4423.55 \text{ gained}}$ ← This represents the total gain to producers who are willing to supply units at a lower price than $\bar{p} = \$34.03$ but are still able to supply units at $\bar{p}$.

3. Acme Media Company has a uniform annual demand for 21,600 DVDs. It costs \$0.50 to store a DVD for one year and \$294 to set up the machinery that produces their DVDs. How many times per year should Acme produce DVDs to minimize the total storage and setup costs?

Iventory Control : Let x = the number of DVDs produced in each production run.
Let y = the number of production runs per year.

$\boxed{\text{Minimize Costs}}$ $C(x) = 0.50\left(\frac{x}{2}\right) + 294y$ on $[1, 21600]$

$xy = 21600$

$y = \frac{21600}{x}$

$C(x) = 0.25x + 294(21600x^{-1})$

$\boxed{C(x) = 0.25x + 6{,}350{,}400x^{-1} \text{ on } [1, 21600]}$

$C'(x) = 0.25 - 6350400x^{-2}$

$= \frac{0.25x^2 - 6350400}{x^2} = 0$

$0.25x^2 - 6350400 = 0$

$x^2 = 25401600$

$x = \pm 5040$

(take $x = 5040$ only)

$y = \frac{21600}{5040}$

$y = 4.2857$

Check

x	$C(x)$
1	\$6350400.25
5040	\$2520 ← min
21600	\$5694

Acme should produce DVDs about 4 times per year to minimize costs

4. Find the absolute extrema of $f(x) = (5-x)(x+7)^2$ on

(a) $[-9, -5]$

$f'(x) = (-1)(x+7)^2 + (5-x)(2)(x+7)^1(1)$

$= (x+7)[-1(x+7) + 2(5-x)]$

$= (x+7)(-x-7+10-2x)$

$= (x+7)(-3x+3) = 0$

$x = -7$ $\quad -3x+3=0$

$-3x = -3$

$x = 1$

Closed Interval Method

x	$f(x)$
-9	56 ← abs. max
-7	0 ← abs. min
-5	40

On the interval $[-9,-5]$, the absolute maximum value of $f(x)$ is 56 and occurs at $x=-9$, and the absolute minimum value of $f(x)$ is 0 and occurs at $x=-7$.

(b) $(0, \infty)$

$x=1$ is the only critical number in this interval, so use 2nd Deriv. Test.

$f'(x) = (x+7)(-3x+3)$

$f''(x) = (1)(-3x+3) + (x+7)(-3)$

$f''(x) = -3x+3-3x-21$

$f''(x) = -6x-18$

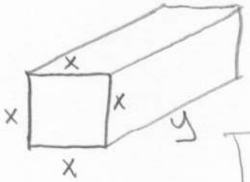
$f''(1) = -6-18 = -24 < 0$ ← abs. max

$f(1) = 256$

On the interval $(0,\infty)$, the absolute maximum value of $f(x)$ is 256 and occurs at $x=1$. $f(x)$ has no absolute min. on this interval.

5. The U.S. Postal Service considers a package to be regular-sized if its length plus girth (distance around) does not exceed 84 inches.

   (a) Find the dimensions of a rectangular box with square [redacted] ends that satisfies this restriction and has maximum volume. What is the maximum volume?



$$\text{Girth} = 4x + y = 84 \text{ in} \Rightarrow y = 84 - 4x$$
$$\text{Volume} = x^2 y = V(x)$$
$$\boxed{\text{maximize Volume } V(x) = x^2(84-4x) \text{ on } [0, 21]}$$

Domain

$x \geq 0$

$y \geq 0$

$\downarrow$

$84 - 4x \geq 0$

$-4x \geq -84$

$x \leq 21$

$$V(x) = 84x^2 - 4x^3$$
$$V'(x) = 168x - 12x^2 = 0$$
$$12x(14 - x) = 0$$
$$12x = 0 \qquad 14 - x = 0$$
$$x = 0 \qquad x = 14$$

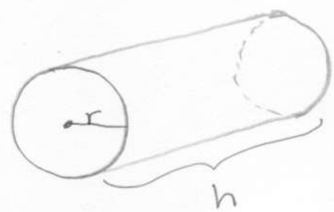
Closed Interval Method

x	$V(x)$
0	0
14	5488 ← abs. max
21	0

$x = 14$ so $y = 84 - 4(14) = 28$

The box with maximum volume has dimensions 14 in × 14 in × 28 in. The maximum volume is 5488 in^3.

   (b) Find the dimensions (radius and height) of a cylindrical container that satisfies this restriction and has maximum volume. What is the maximum volume?



$$\text{Girth} = 2\pi r + h = 84 \Rightarrow h = 84 - 2\pi r$$
$$\text{volume} = \pi r^2 h = V(r)$$
$$\boxed{\text{maximize } V(r) = \pi r^2(84 - 2\pi r) \text{ on } [0, \tfrac{42}{\pi}]}$$

Domain

$r \geq 0$

$h \geq 0$

$\downarrow$

$84 - 2\pi r \geq 0$

$-2\pi r \geq -84$

$r \leq \frac{42}{\pi} \approx 13.3690$

$$V(r) = 84\pi r^2 - 2\pi^2 r^3$$
$$V'(r) = 168\pi r - 6\pi^2 r^2 = 0$$
$$6\pi r(28 - \pi r) = 0$$
$$6\pi r = 0 \qquad 28 - \pi r = 0$$
$$r = 0 \qquad -\pi r = -28$$
$$r = \frac{28}{\pi} \approx 8.9127$$

$r = \frac{28}{\pi} \Rightarrow h = 84 - 2\pi\left(\frac{28}{\pi}\right)$

$h = 28$

Closed Interval Method

r	$V(r)$
0	0
$\frac{28}{\pi}$	4987.5386 ← abs. max
$\frac{42}{\pi}$	0

The dimensions of the cylindrical container with maximum volume are radius $= \frac{28}{\pi}$ inches and height $= 28$ in. The maximum volume is 4987.5386 in^3.

6. Find the equation of the curve that pases through the point $(1,9)$ if its slope is given by $\frac{dy}{dx} = 8x^3 - 5x^{-1} + 4$ for all $x \neq 0$.

$y = \int (8x^3 - 5x^{-1} + 4)\,dx$

$y = 8\left(\frac{1}{4}x^4\right) - 5\ln|x| + 4x + C$

$y = 2x^4 - 5\ln|x| + 4x + C$

$y(1) = 2(1)^4 - 5\ln|1| + 4(1) + C = 9$

$2 + 4 + C = 9$

$C = 3$

$\boxed{y = 2x^4 - 5\ln|x| + 4x + 3}$

7. Compute each of the following by hand.

TYPO! $\longrightarrow$ $10x$

(a) $\int (e^{\bullet} - x)(e^{10x} - 5x^2)\,dx$

$u = e^{10x} - 5x^2$

$du = (10e^{10x} - 10x)\,dx$

$= \frac{1}{10}\int (e^{10x} - 5x^2)(10e^{10x} - 10x)\,dx$

$= \frac{1}{10}\int u\,du$

$= \frac{1}{10}\left(\frac{u^2}{2}\right) + C = \boxed{\frac{1}{20}(e^{10x} - 5x^2)^2 + C}$

(b) $\int \frac{\ln x + 7}{x}\,dx$

$u = \ln x + 7$

$du = \frac{1}{x}\,dx$

$= \int (\ln x + 7)\left(\frac{1}{x}\right)dx$

$= \int u\,du$

$= \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\ln x + 7)^2 + C}$

(c) $\int \left(3\sqrt{t} + \frac{5}{\sqrt[3]{t^8}} + \frac{6}{t} - 1\right) dt$

$= \int \left(3t^{1/2} + 5t^{-\frac{8}{3}} + 6\left(\frac{1}{t}\right) - 1\right) dt$

$= 3\left(\frac{2}{3}t^{3/2}\right) + 5\left(-\frac{3}{5}t^{-5/3}\right) + 6\ln|t| - t + C$

$= \boxed{2t^{\frac{3}{2}} - 3t^{-5/3} + 6\ln|t| - t + C}$

(d) $\int_1^2 m^2(m^4 - 7m^{-6})dm$

$$= \int_1^2 (m^6 - 7m^{-4})\,dm$$

$$= \left(\frac{m^7}{7} - 7\left(\frac{m^{-3}}{-3}\right)\right)\Big|_1^2$$

$$= \left(\tfrac{1}{7}m^7 + \tfrac{7}{3}m^{-3}\right)\Big|_1^2$$

$$= \left(\tfrac{1}{7}(2)^7 + \tfrac{7}{3}(2)^{-3}\right) - \left(\tfrac{1}{7}(1)^7 + \tfrac{7}{3}(1)^{-3}\right)$$

$$= \frac{3121}{168} - \frac{52}{21} = \boxed{\frac{2705}{168}}$$

(e) $\int_0^{\frac{1}{7}} \frac{e^{7x}}{5 - e^{7x}}dx$

$u = 5 - e^{7x}$

$du = -7e^{7x}dx$

when $x = 0$, $u = 5 - e^{7(0)} = 5 - 1 = 4$

when $x = \frac{1}{7}$, $u = 5 - e^{7(\frac{1}{7})}$

$u = 5 - e$

$$= -\frac{1}{7}\int_0^{\frac{1}{7}} \frac{-7e^{7x}}{5 - e^{7x}}dx$$

$$= -\frac{1}{7}\int_4^{5-e} \frac{1}{u}\,du$$

$$= -\frac{1}{7}\ln|u|\Big|_4^{5-e}$$

$$= \left(-\tfrac{1}{7}\ln|5-e|\right) - \left(-\tfrac{1}{7}\ln 4\right) = -\tfrac{1}{7}\ln|5-e| + \tfrac{1}{7}\ln 4 \approx \boxed{0.0802}$$

(f) $\int z(z+4)^8 dz$

$u = z + 4 \rightarrow z = u - 4$

$du = dz$

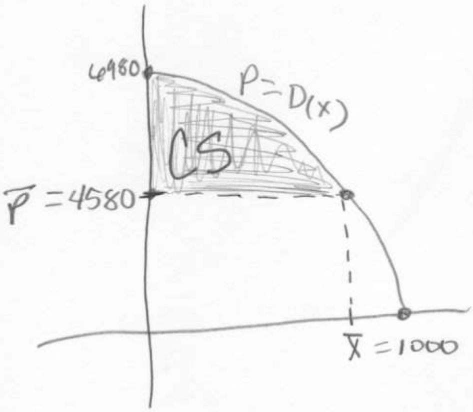
$$= \int z(u)^8 du$$

$$= \int (u-4)u^8 du$$

$$= \int (u^9 - 4u^8)du$$

$$= \tfrac{1}{10}u^{10} - \tfrac{4}{9}u^9 + C = \boxed{\tfrac{1}{10}(z+4)^{10} - \tfrac{4}{9}(z+4)^9 + C}$$

8. The price-demand equation for a certain item is given by $p = D(x) = -0.002(x+100)^2 + 7000$ dollars per item, where x is the number of items that can be sold at a price of \$$p$. If the current price per item is \$4,580, find the consumers' surplus. What does this number represent?



$$CS = \int_0^{\bar{x}} (D(x) - \bar{p})\,dx$$

$$= \int_0^{1000} (Y_1 - 4580)\,dx$$

$$= \text{fnInt}(Y_1 - 4580, X, 0, 1000)$$

$$= \boxed{\$1{,}533{,}333.33 \text{ saved}}$$

To find $\bar{x}$:

$Y_1 = -0.002(X+100)^2 + 7000$

$Y_2 = 4580$

Graph, calc. Intersect

$X = 1000 \quad y = 4580$

This represents the total savings to consumers who are willing to pay more than \$4,580 per item but are still able to buy the product for \$4,580.

9. Acme Furniture Company's marginal cost for its dinette sets is given by $C'(x) = 3e^{0.01x} + \dfrac{375}{\sqrt{x}}$ dollars per set, where x is the number of dinette sets produced each month.

(a) Find the change in total cost that results from going from a production level of 100 to 150 dinette sets per month. → equal to area under its deriv.

$$\text{Change in cost} = \int_{100}^{150} \left(3e^{0.01x} + \frac{375}{\sqrt{x}}\right) dx$$

$$= \text{fnInt}(3e^{0.01x} + 375/\sqrt{x}, X, 100, 150)$$

$$= \boxed{\$2214.61}$$

(b) If Acme's fixed cost for producing dinette sets is \$1,700, find a model for total cost.

$$C(x) = \int \left(3e^{0.01x} + 375x^{-\frac{1}{2}}\right) dx$$

$$C(x) = \frac{3}{0.01} e^{0.01x} + 375\left(\frac{2}{1}x^{\frac{1}{2}}\right) + C$$

$$C(x) = 300e^{0.01x} + 750\sqrt{x} + C$$

$$C(0) = 300e^0 + 750\sqrt{0} + C = 1700$$

$$300 + C = 1700$$

$$C = 1400$$

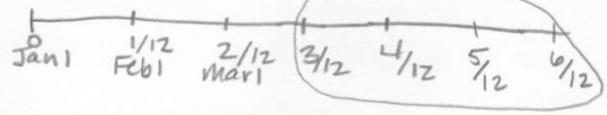
$$\boxed{C(x) = 300e^{0.01x} + 750\sqrt{x} + 1400 \text{ dollars, where } x \text{ is the number of dinette sets produced each month.}}$$

10. Bob invested \$3,000 into an account paying 7.3% per year compounded continuously at the beginning of 2003. Find the average account balance during the second quarter of 2003.

2nd 3-month period

$A(t) = Pe^{rt}$

$A(t) = 3000e^{0.073t}$

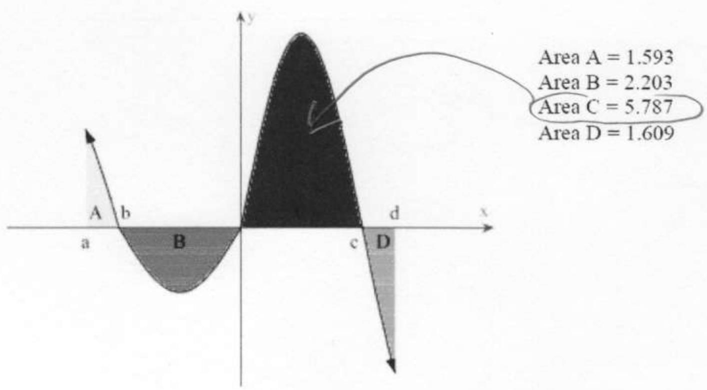


$$\text{Avg value} = \frac{1}{\frac{6}{12} - \frac{3}{12}} \int_{3/12}^{6/12} \left(3000e^{0.073t}\right) dt = \text{4fnInt}\left(3000e^{0.073t}, t, \tfrac{3}{12}, \tfrac{6}{12}\right)$$

reduces to 4

$$= \boxed{\$3083.30}$$

11. Calculate the following definite integrals by referring to the graph of $f(x)$ and indicated area below. (courtesy Jenn Whitfield)



(a) $\int_b^0 f(x)dx = \boxed{-2.203}$

(b) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^0 f(x)dx + \int_0^c f(x)dx$

$= 1.593 + (-2.203) + 5.787 = \boxed{5.177}$

(c) $\int_b^d f(x)dx = -2.203 + 5.787 - 1.609$

$= \boxed{1.975}$

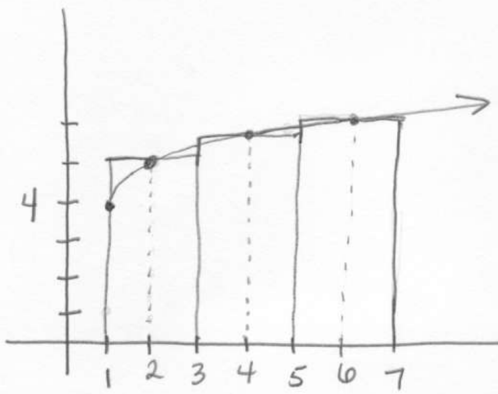
(d) $\int_d^c f(x)dx = -\int_c^d f(x)dx$

$= -(-1.609)$

$= \boxed{1.609}$

12. Approximate the area under $f(x) = \sqrt{x-1}+4$ on the interval $[1,7]$ using a midpoint sum with 3 rectangles of equal width. Include an appropriate sketch with your answer.

$\text{width} = \frac{7-1}{3} = \frac{6}{3} = 2$



Method 1

$\text{Area} = 2(f(2)+f(4)+f(6))$

$= 2(5 + 5.7321 + 6.2361)$

$= \boxed{33.9364}$

Method 2

Midpoint	Height	Width	Area
2	$f(2)=5$	2	$2*5=10$
4	$f(4)=5.7321$	2	11.4642
6	$f(6)=6.2361$	2	+12.4722
			$\boxed{33.9364}$

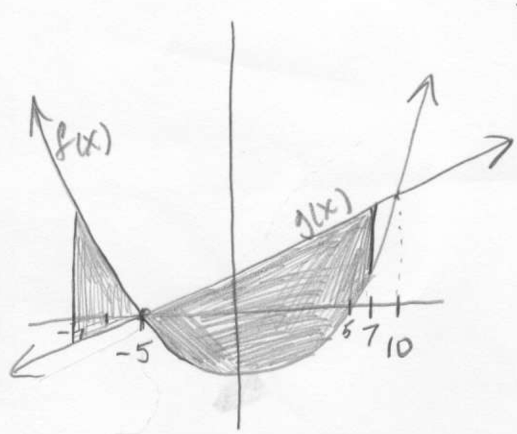
13. If $f(x)$ in the previous problem represents the rate at which a swimming pool is filling with water and is given in gallons per minute, where x is the number of minutes since 5:30pm, what does the approximated area above represent?

height * width

(gallons perminute) * (minutes) = gallons

From 5:31pm to 5:37pm, the volume of water in the swimming pool increased by about 33.9364 gallons.

14. Find the area between $\underbrace{f(x) = 0.04x^2 - 1}_{Y_1}$ and $\underbrace{g(x) = 0.2x+1}_{Y_2}$ on the interval $[-7,7]$.



graph, 2nd calc. Intersect
(not shown to scale)

$\text{Area} = \int_{-7}^{-5}(f(x)-g(x))\,dx + \int_{-5}^{7}(g(x)-f(x))\,dx$

$= \text{fnInt}(Y_1 - Y_2, X, -7, -5) + \text{fnInt}(Y_2 - Y_1, X, -5, 7)$

$= \frac{98}{75} + \frac{504}{25}$

$= \boxed{\frac{322}{15}}$

15. Given that $f(x) = \dfrac{3x^2+2}{x^2-25}$, $f'(x) = \dfrac{-154x}{(x^2-25)^2}$, and $f''(x) = \dfrac{462x^2+3850}{(x^2-25)^3}$,

(a) Find the domain of $f(x)$.

$x^2-25=0$

$x=\pm 5$

Domain: $(-\infty,-5)\cup(-5,5)\cup(5,\infty)$

(all real numbers except $x=\pm 5$)

(b) Find all intercepts of $f(x)$.

x-ints

$f(x)=0 \Rightarrow 3x^2+2=0$

$3x^2=-2$

$x^2=-\frac{2}{3}$

no real soln.

$\boxed{\text{no x-intercepts}}$

y-int:

$f(0)=\frac{0+2}{0-25}=\frac{-2}{25}$

$\boxed{\text{y-int} = (0, -\frac{2}{25})}$

(c) Find all asymptotes of $f(x)$.

vertical: factor everything first!

$f(x)=\dfrac{3x^2+2}{(x+5)(x-5)}$

$\boxed{\text{Vertical asymptotes: } x=-5 \text{ and } x=5}$

Horizontal asymptote:

$\lim\limits_{x\to\infty} \dfrac{3x^2+2}{x^2-25} = \frac{3}{1}=3$

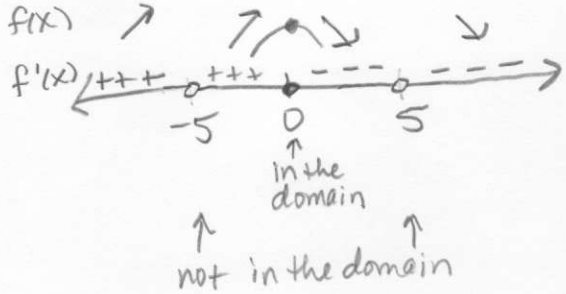
$\boxed{y=3 \text{ is horiz. asymptote}}$

(d) Find all intervals where $f(x)$ is increasing and all intervals where $f(x)$ is decreasing.

$f'(x)=0$ only if $-154x=0$

$x=0$

$f'(x)=\dfrac{-154x}{(x^2-25)^2}$ DNE for $x=\pm 5$



Test#	$f'(x)$
-6	$\frac{84}{11}$
-1	$\frac{77}{288}$
1	$-\frac{77}{288}$
6	$-\frac{84}{11}$

• $f(x)$ is increasing on $(-\infty,-5)$ and $(-5,0)$.

• $f(x)$ is decreasing on $(0,5)$ and $(5,\infty)$.

(e) Find the coordinates of all local extrema.

local maximum at $x=0$, $f(0)=-\frac{2}{25}$

$(0, -\frac{2}{25})$

(f) Find all intervals where $f(x)$ is concave upward and all intervals where $f(x)$ is concave downward.

$f''(x) = 0$ only if $462x^2 + 3850 = 0$

$462x^2 = -3850$

$x^2 = -\frac{25}{3}$

no real soln.

$f''(x) = \dfrac{462x^2 + 3850}{(x^2-25)^3}$ DNE for $x = \pm 5$

$f''(x)$: +++ | − − − | +++ at -5, 5 (not in domain)

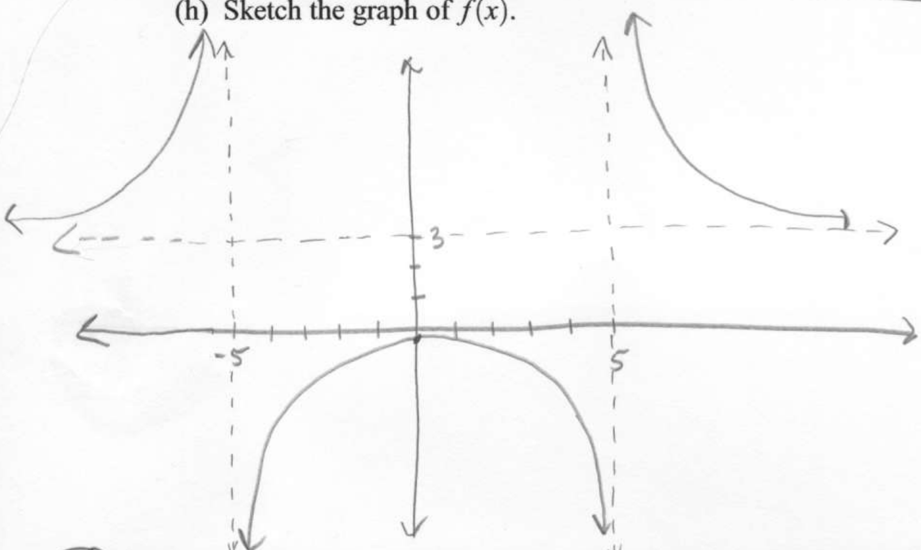
Test #	$f''(x)$
-6	$1862/121$
0	$-154/625$
6	$1862/121$

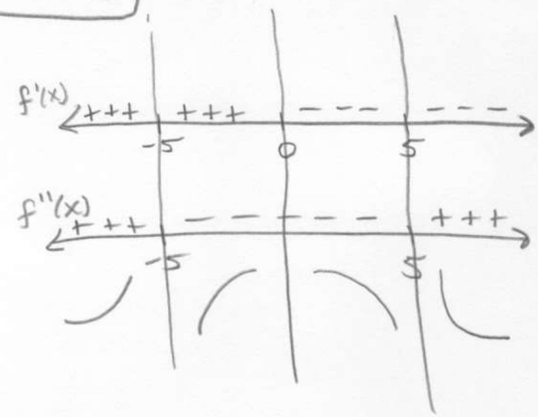
- $f(x)$ is concave upward on $(-\infty, -5)$ and $(5, \infty)$.
- $f(x)$ is concave downward on $(-5, 5)$.

(g) Find the coordinates of all inflection points.

Since $x = \pm 5$ are not in the domain, $f(x)$ has $\boxed{\text{no inflection points}}$

(h) Sketch the graph of $f(x)$.





16. Suppose that x is a continuous random variable with associated probability density function

This material was not covered in all classes. Ask your instructor.

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that x is less than 1.5.

less than 1.5 means we want $P(1 \leq X < 1.5)$

$$P(1 \leq X < 1.5) = \int_1^{1.5} \frac{2}{x^3}\,dx$$

$$= \text{fnInt}(2/x^3, X, 1, 1.5) = \boxed{\frac{5}{9}}$$