Math 142 - Final Exam Review

NOTE: This review is intended to highlight the material covered on the Final Exam but should not be used as your sole source of practice. Also refer to your instructor's lecture notes, previous week-in-reviews, previous exams and quizzes, suggested homework, supplemental homework, and the online homework as additional sources for review and exam preparation.

1. Find the locations of all local extrema and saddle points of each of the following functions.

(a) $f(x,y) = -0.5x^2 + 5xy - 2y^2 + 10x + 13y + 1$

$f_x(x,y) = -x + 5y + 10 = 0$
$x = 5y + 10$

$f_y(x,y) = 5x - 4y + 13 = 0$
$5(5y+10) - 4y + 13 = 0$
$25y + 50 - 4y + 13 = 0$
$21y = -63$
$y = -3$
$x = 5(-3) + 10$
$x = -5$
$(-5,-3)$ is a critical point

$f_{xx}(x,y) = -1$ $\quad f_{yy}(x,y) = -4$
$f_{xy}(x,y) = 5 \overset{\checkmark}{=} f_{yx}(x,y) = 5$

$A = -1$
$B = 5$
$C = -4$

$AC - B^2 = (-1)(-4) - (5)^2 = -21 < 0$

$\boxed{f(x,y) \text{ has a saddle point at } (-5,-3).}$

(b) $g(x,y) = 2x^2 - 2x^2y + 6y^3$

$g_x(x,y) = 4x - 4xy = 0$
$4x(1-y) = 0$
$4x = 0 \quad 1 - y = 0$
$x = 0 \quad y = 1$

$g_y(x,y) = -2x^2 + 18y^2 = 0$
when $x = 0$,
$0 + 18y^2 = 0$
$y = 0$
$(0,0)$ is a critical point.

when $y = 1$, $-2x^2 + 18(1)^2 = 0$
$-2x^2 = -18$
$x^2 = 9$
$x = \pm 3$
$(3,1)$ and $(-3,1)$ are critical pts.

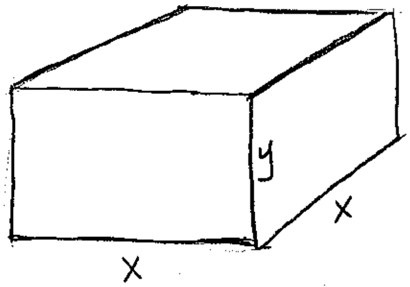
$g_{xx}(x,y) = 4 - 4y$ $\quad g_{yy}(x,y) = 36y$
$g_{xy}(x,y) = -4x \overset{\checkmark}{=} g_{yx}(x,y) = -4x$

$(0,0)$
$A = g_{xx}(0,0) = 4 - 4(0) = 4$
$B = g_{xy}(0,0) = -4(0) = 0$
$C = g_{yy}(0,0) = 36(0) = 0$
$AC - B^2 = 4(0) - 0^2 = 0$
$\boxed{\text{Test Fails}}$

$(3,1)$
$A = g_{xx}(3,1) = 4 - 4(1) = 0$
$B = g_{xy}(3,1) = -4(3) = -12$
$C = g_{yy}(3,1) = 36(1) = 36$
$AC - B^2 = 0(36) - (-12)^2 = -144 < 0$
$g(x,y)$ has a saddle point at $(3,1)$.

$(-3,1)$
$A = g_{xx}(-3,1) = 4 - 4(1) = 0$
$B = g_{xy}(-3,1) = -4(-3) = 12$
$C = g_{yy}(-3,1) = 36(1) = 36$
$AC - B^2 = 0(36) - 12^2 = -144 < 0$
$g(x,y)$ has a saddle point at $(-3,1)$.

2. A rectangular box is to have a square base and a volume of 20 cubic feet. If the material for the base costs 30 cents per square foot, the material for the sides costs 10 cents per square foot, and the material for the top costs 20 cents per square foot, determine the dimensions of the box that can be constructed at minimum cost. (source: #8, pg. 365 of *Applied Calculus for the Managerial, Life, and Social Sciences*, 5th ed., by Tan)



Volume $= x^2 y = 20 \longrightarrow y = \frac{20}{x^2}$

$$\text{Cost} = C(x) = 0.3x^2 + 0.1(4xy) + 0.2x^2$$
$$= 0.5x^2 + 0.4xy$$
$$= 0.5x^2 + 0.4x\left(\frac{20}{x^2}\right)$$
$$= 0.5x^2 + 8x^{-1}$$

$\boxed{\text{Minimize } C(x) = 0.5x^2 + 8x^{-1} \text{ on } (0, \infty)}$

$$C'(x) = x - 8x^{-2} = 0$$
$$\left(\frac{x^2}{x^2}\right)x - \frac{8}{x^2} = 0$$
$$\frac{x^3 - 8}{x^2} = 0$$
$$x^3 - 8 = 0$$
$$x^3 = 8$$
$$x = 2$$

$$C''(x) = 1 + 16x^{-3}$$
$$C''(2) = 1 + \frac{16}{2^3} = 3 > 0 \quad \text{min}$$

$x = 2$

$y = \frac{20}{2^2} = 5$

$\boxed{\text{To minimize costs, the box should be 2 ft by 2 ft by 5 ft.}}$

3. What sequence of graph transformations must be performed to obtain the graph of $h(x) = -5\,|\,x+1\,| - 6$ from the graph of $f(x) = |\,x\,|$?

① Shift $|x|$ one unit to the left.
② Stretch vertically by a factor of 5.
③ Reflect about the x-axis.
④ Shift down 6 units.

4. Find the $\underbrace{\text{instantaneous rate of change}}_{\text{derivative}}$ of $g(x) = \ln(7x - 5)$ at $x = 2$.

$$g'(x) = \frac{1}{7x-5}(7)$$

$$g'(2) = \frac{7}{7(2) - 5} = \boxed{\frac{7}{9}}$$

5. The table below shows the population of a city in China for different years.

Year	1930	1940	1950	1960	1970	1980
Population (in thousands)	390	700	900	1000	950	830

(a) Find the best model $P(t)$ for the data, assuming that the model will be used for interpolation purposes only. Let t represent the number of years since 1930.

$P(t) = -0.5625t^2 + 36.8393t + 389.6429$ thousand people, where t is the number of years since 1930.

(b) Use your model to estimate when the population of the city reached 750,000 people.

$P(t) \rightarrow Y_1$ Calc. Intersect. $t = 11.9694$ and $t = 53.5226$

$750 \rightarrow Y_2$ ≈ 1942 ≈ 1984

The population reached 750,000 people in 1942 and 1984.

6. Compute each of the following by hand.

(a) $\int_1^a (7x^{-1} + x^{0.2})dx$ where $a > 1$

$$= \left(7\ln|x| + \frac{x^{1.2}}{1.2}\right)\Big|_1^a$$

$$= \left(7\ln a + \frac{5}{6}a^{1.2}\right) - \left(7\ln 1 + \frac{5}{6}(1)^{1.2}\right)$$

$$= \boxed{7\ln a + \frac{5}{6}a^{1.5} - \frac{5}{6}}$$

(b) $\int_9^{21} (t+1)\sqrt{t-5}dt$

$u = t - 5 \longrightarrow t = u + 5$

$du = dt$

when $t = 9$, $u = 9 - 5 = 4$

when $t = 21$, $u = 21 - 5 = 16$

$$= \int_4^{16} (u+5+1)\sqrt{u}\, du$$

$$= \int_4^{16} (u+6)u^{1/2} du$$

$$= \int_4^{16} (u^{\frac{3}{2}} + 6u^{\frac{1}{2}}) du$$

$$= \left(\frac{2}{5}u^{\frac{5}{2}} + 6\left(\frac{2}{3}u^{3/2}\right)\right)\Big|_4^{16} = \left(\frac{2}{5}(16)^{\frac{5}{2}} + 4(16)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(4)^{\frac{5}{2}} + 4(4)^{\frac{3}{2}}\right)$$

$$= 665.6 - 44.8 = \boxed{620.8}$$

7. Find the domain of each of the following functions.

(a) $f(x) = \dfrac{\ln(x-7)}{\sqrt[3]{x^2-16}}$

$\boxed{\text{Domain: } (7, \infty)}$

$x - 7 > 0$ AND $x^2 - 16 \neq 0$

$x > 7$

$x^2 \neq 16$

$x \neq \pm 4$

(neither are greater than 7)

(b) $g(x) = \begin{cases} \sqrt[8]{10-x} & \text{if } x \leq 20 \\ \dfrac{1}{x^3 - 38x^2 - 80x} & \text{if } x > 20 \end{cases}$

On $(-\infty, 20]$

$10 - x \geq 0$

$-x \geq -10$

$x \leq 10$

on $(20, \infty)$

$x^3 - 38x^2 - 80x \neq 0$

$x(x^2 - 38x - 80) \neq 0$

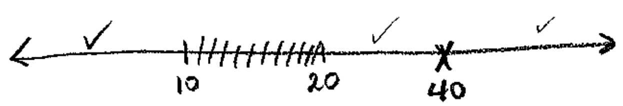
$x(x-40)(x+2) \neq 0$

$x \neq 0$

$x \neq 40 \leftarrow$ in interval

$x \neq -2$

$\boxed{\text{Domain: } (-\infty, 10] \cup (20, 40) \cup (40, \infty)}$



8. Bob purchased a tractor for \$27,500 in 1980 and 7 years later, it was worth only \$14,788. Assuming linear depreciation, find when the tractor will be worth \$5,700.

$(0, 27500)$ x = # of years since 1980

$(7, 14788)$ y = value of tractor

L_1	L_2
0	27500
7	14788

LinReg(ax+b) L_1, L_2

$y = -1816x + 27500$

$5700 = -1816x + 27500$

$-21800 = -1816x$

$12.0044 = x$

The tractor will be worth \$5,700 in 1992.

9. Find the equation of $g(x)$ if its graph is obtained by taking the graph of $f(x) = e^x$ and shifting it 8 units to the right, stretching it vertically by a factor of 7, and then shifting it down 2 units.

$g(x) = 7e^{x-8} - 2$

10. C&G Imports, Inc., imports two brands of white wine, one from Germany and the other from Italy. The German wine costs \$4 per bottle, and the Italian wine can be obtained for \$3 per bottle. It has been estimated that if the German wine retails at p dollars per bottle and the Italian wine is sold for q dollars per bottle, then $x = 2000 - 150p + 100q$ bottles of German wine and $y = 1000 + 80p - 120q$ bottles of Italian wine will be sold per week. (source: #24, pg. 635 of *Applied Calculus for the Managerial, Life, and Social Sciences*, 5th ed., by Tan)

(a) Find the weekly revenue function $R(p,q)$.

$$R(p,q) = px + qy = p(2000 - 150p + 100q) + q(1000 + 80p - 120q)$$
$$\boxed{R(p,q) = 2000p - 150p^2 + 180pq + 1000q - 120q^2}$$

(b) Find the weekly profit function $P(p,q)$.

$$P(p,q) = R(p,q) - C(p,q) \text{ where } C(p,q) = 4x + 3y$$
$$= 2000p - 150p^2 + 180pq + 1000q - 120q^2 - [4(2000 - 150p + 100q) + 3(1000 + 80p - 120q)]$$
$$\boxed{P(p,q) = 2360p - 150p^2 + 180pq + 960q - 120q^2 - 11000 \text{ dollars}}$$

(c) Evaluate and interpret $P_q(15,12)$

$P_q(p,q) = 180p + 960 - 240q$ dollars per dollar/bottle of Italian wine.

$P_q(15,12) = 180(15) + 960 - 240(12) = \$780$ per dollar/bottle of Italian wine.

When the price per bottle of Italian wine is \$12, for every \$1 increase in price per bottle of the Italian wine, weekly profit will increase by \$780, assuming the price per bottle of German wine remains constant at \$15 per bottle.

(d) Determine the unit price for each brand that will allow C&G to realize the largest possible weekly profit.

$P_p(p,q) = 2360 - 300p + 180q = 0$

$180q = 300p - 2360$

$q = \frac{5}{3}p - \frac{118}{9}$

$P_q(p,q) = 180p + 960 - 240q = 0$

$180p - 240\left(\frac{5}{3}p - \frac{118}{9}\right) = -960$

$-220p + \frac{9440}{3} = -960$

$-220p = -\frac{12320}{3}$

$p = \frac{56}{3} \approx \$18.67$

$q = \frac{5}{3}\left(\frac{56}{3}\right) - \frac{118}{9} = \$18$

Critical point: $\left(\frac{56}{3}, 18\right)$

$P_{pp}(p,q) = -300$ $\quad P_{qq}(p,q) = -240$

$P_{pq}(p,q) = 180 \overset{\checkmark}{=} P_{qp}(p,q) = 180$

$A = -300$
$B = 180$
$C = -240$

$AC - B^2 = (-300)(-240) - 180^2$

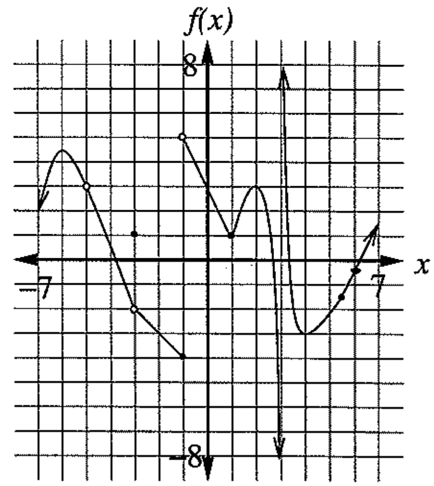
$= 39600 > 0$

Now check $A = f_{xx}\left(\frac{56}{3}, 18\right)$:

$A = -300 < 0$ $\wedge$ max

For maximum weekly profit, the German wine should be sold at \$18.67/bottle and the Italian for \$18/bottle.

11. Refer to the graph of $f(x)$ to find each of the following:



(a) $\lim_{x\to 6^+} f(x) \approx -0.4$

(c) $\lim_{x\to 3^-} f(x) = -\infty$

(b) $\lim_{x\to -1^+} f(x) = 5$

(d) $\lim_{x\to -3} f(x) = -2$

$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = -2$

(e) Find all x for which $f(x)$ is discontinuous on the interval $(-7,5)$.

$f(x)$ is discontinuous at $x = -5, -3, -1, 3$ on $(-7,5)$.

12. Find the equation of the tangent line to $f(x) = 0.5e^{2x}$ at $x = 1$.

point: $x = 1$ $\quad y = f(1) = 0.5e^{2(1)} = 0.5e^2$ $\quad (1, 0.5e^2)$

$m = f'(1) = e^2$

$f'(x) = 0.5e^{2x}(2)$

$f'(x) = e^{2x}$

$f'(1) = e^{2(1)} = e^2$

$y - 0.5e^2 = e^2(x-1)$

$y = e^2x - e^2 + 0.5e^2$

$\boxed{y = e^2x - 0.5e^2}$

13. Find the absolute extrema of $f(x) = xe^{0.2x}$ on the interval $[-6, 1]$.

$f'(x) = (1)e^{0.2x} + xe^{0.2x}(0.2) = 0$

$= e^{0.2x}(1 + 0.2x) = 0$

$e^{0.2x} = 0$ No solution

$1 + 0.2x = 0$
$0.2x = -1$
$x = -5$

x	$f(x) = xe^{0.2x}$
-6	$-6e^{0.2(-6)} = -6e^{-1.2} \approx -1.8072$
-5	$-5e^{0.2(-5)} = -5e^{-1} \approx -1.8394$
1	$e^{0.2} \approx 1.2214$

On the interval $[-6,1]$, the absolute minimum value of $f(x)$ is -1.8394 and occurs at $x = -5$, and the absolute maximum value is 1.2214 and occurs at $x = 1$.

14. Find each of the following limits, if they exist.

(a) $\lim_{x\to-3}\frac{x^2+8x+15}{x^2+x-6}$ $= \lim_{x\to-3}\frac{\cancel{(x+3)}(x+5)}{\cancel{(x+3)}(x-2)}$

$= \lim_{x\to-3}\frac{x+5}{x-2} = \frac{-3+5}{-3-2} = \boxed{-\frac{2}{5}}$

(b) $\lim_{x\to0}\frac{2x^2-7}{5x+14} = \frac{2(0)^2-7}{5(0)+14} = \boxed{-\frac{1}{2}}$

(c) $\lim_{x\to\infty}\frac{4x^2-3x^3+2}{5-7x+12x^2} = \lim_{x\to\infty}\frac{-3x^3}{12x^2} = \lim_{x\to\infty}-\frac{1}{4}x = \boxed{-\infty}$

(d) $\lim_{x\to-\infty}\frac{7x^4+5x}{10x^3-21x^4} = \frac{7}{-21} = \boxed{-\frac{1}{3}}$

15. Find all asymptotes and x-coordinates of any holes for $f(x) = \frac{5x^2-10x-15}{(x+2)^2(x^2-3x)}$

$f(x) = \frac{5(x^2-2x-3)}{(x+2)^2\,x(x-3)} = \frac{5\cancel{(x-3)}(x+1)}{x\cancel{(x-3)}(x+2)^2}$

$\lim_{x\to\infty}\frac{5x^2-10x-15}{(x+2)^2(x^2-3x)} = 0$

(higher degree in denom.)

hole at $x=3$

vertical asymptotes: $x=0$ and $x=-2$

horizontal asymp: $y=0$

16. Find $\lim_{x \to -\infty} (-6x^3 + 3x - 7)$.

odd degree with negative leading coefficient, so

$$\lim_{x \to -\infty} (-6x^3 + 3x - 7) = \boxed{\infty}$$

17. Bob runs a tutoring service. He prices each tutoring session as follows: For a group of 20 or more people, he charges the group a flat fee of \$200 plus \$10 per person. For a group of 8 to 19 people, he charges the group a flat fee of \$125 plus \$15 per person. For a session with fewer than 8 people, he charges a flat fee of \$55 plus \$20 per person. Write a piecewise-defined function to represent the total bill for one session with a group of x people.

$$f(x) = \begin{cases} 55 + 20x & \text{if } 0 \le x \le 7 \\ 125 + 15x & \text{if } 8 \le x \le 19 \\ 200 + 10x & \text{if } x \ge 20 \end{cases}$$

dollars, where x is an integer representing the number of people in the group.

18. Use the definition of the derivative to find $f'(x)$ for $f(x) = x - 9x^2$.

① $f(x+h) = (x+h) - 9(x+h)^2$
$= x + h - 9(x^2 + 2xh + h^2)$
$= x + h - 9x^2 - 18xh - 9h^2$

② $f(x+h) - f(x) = \cancel{x} + h - \cancel{9x^2} - 18xh - 9h^2 - (\cancel{x} - \cancel{9x^2})$
$= h - 18xh - 9h^2$

③ $$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{h}(1 - 18x - 9h)}{\cancel{h}}$$
$= 1 - 18x - 9h$

④ $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (1 - 18x - 9h) = 1 - 18x$$

so $\boxed{f'(x) = 1 - 18x}$

19. Acme Music Company has determined the price-demand function for its home audio system to be $p = 3080 - 10x$ dollars per system. Acme Music has fixed costs that amount to \$56,250 and variable costs of \$580 per system.

(a) How many home audio systems must be sold to maximize revenue? Use two methods to obtain your answer.

$R(x) = px = 3080x - 10x^2$

Method 1: Algebra

$R(x) = -10x^2 + 3080x$

This is a parabola opening downward, so the maximum occurs at the vertex.

$x = \dfrac{-3080}{2(-10)} = \boxed{154 \text{ home audio systems}}$

Method 2: Calculus

$R(x) = -10x^2 + 3080x$

$R'(x) = -20x + 3080 = 0$

$-20x = -3080$

$x = 154$

$R''(x) = -20 < 0$ max

$x = \boxed{154 \text{ home audio systems}}$

(b) Find the profit and marginal profit from the production and sale of 175 home audio systems.

$P(x) = R(x) - C(x)$

$= -10x^2 + 3080x - (580x + 56250)$

$P(x) = -10x^2 + 2500x - 56250$

$\boxed{P(175) = \$75{,}000}$

$P'(x) = -20x + 2500$

$\boxed{P'(175) = -1000 \text{ dollars per system}}$

(c) Use your answers in (b) to approximate the profit from the production and sale of 177 home audio systems.

$P(177) \approx P(175) + 2P'(175)$

$= 75000 + 2(-1000) = \boxed{\$73{,}000}$

(d) How many home audio systems should be produced and sold so that Acme breaks even?

Solve $\underbrace{R(x)}_{Y_1} = \underbrace{C(x)}_{Y_2}$

calc. Intersect

$x = 25$ and $x = 225$

$\boxed{25 \text{ or } 225 \text{ home audio systems}}$

or $P(x) = 0$

$-10x^2 + 2500x - 56250 = 0$

$-10(x^2 - 250x + 5625) = 0$

$-10(x - 225)(x - 25) = 0$

$x = 225$ and $x = 25$

(e) Find the marginal average cost when 100 home audio systems are produced.

$\bar{C}(x) = \dfrac{580x + 56250}{x}$ dollars per system

$\bar{C}(x) = 580 + 56250x^{-1}$

$\bar{C}'(x) = -56250x^{-2}$

$\bar{C}'(100) = \dfrac{-56250}{100^2} = -5.625$

$\approx \boxed{-\$5.63/\text{system per system}}$

20. Find the derivative of each of the following functions. Simplify the function before taking the derivative where possible.

(a) $h(x) = e^{-4.5\ln(x^2+1)}$

$h(x) = e^{\ln(x^2+1)^{-4.5}}$

$h(x) = (x^2+1)^{-4.5}$

$\boxed{h'(x) = -4.5(x^2+1)^{-5.5}(2x)}$

(b) $f(x) = 3x^2 - \frac{5}{\sqrt[4]{x}} + e^{-x} + 7^{2x} - e^{-2}$

$f(x) = 3x^2 - 5x^{-\frac{1}{4}} + e^{-x} + 7^{2x} - e^{-2}$

$\boxed{f'(x) = 6x + \frac{5}{4}x^{-\frac{5}{4}} - e^{-x} + 7^{2x}(2)(\ln 7)}$

(c) $g(x) = \log_8\left((2x-7)\sqrt{x+3}\right)$

$g(x) = \log_8(2x-7) + \log_8(x+3)^{\frac{1}{2}}$

$g(x) = \log_8(2x-7) + \frac{1}{2}\log_8(x+3)$

$\boxed{g'(x) = \left(\frac{1}{\ln 8}\right)\left(\frac{1}{2x-7}\right)(2) + \left(\frac{1}{2}\right)\left(\frac{1}{\ln 8}\right)\left(\frac{1}{x+3}\right)(1)}$

(d) $g(x) = (10x^3 - 1)e^{\sqrt{x}}$ Product Rule!

$\boxed{g'(x) = (30x^2)e^{x^{\frac{1}{2}}} + (10x^3-1)\left(e^{x^{1/2}}\right)\left(\frac{1}{2}x^{-1/2}\right)}$

(e) $m(t) = \left(3\sqrt{t^2+1} - \frac{5}{t}\right)^8$

$$m'(t) = 8\left(3\sqrt{t^2+1} - 5t^{-1}\right)^7 \cdot \frac{d}{dt}\left(3(t^2+1)^{\frac{1}{2}} - 5t^{-1}\right)$$

$$= \boxed{8\left(3\sqrt{t^2+1} - 5t^{-1}\right)^7 \left(\frac{3}{2}(t^2+1)^{-\frac{1}{2}}(2t) + 5t^{-2}\right)}$$

(f) $h(x) = \frac{\ln x + 4x^7}{2x+9}$

$$\boxed{h'(x) = \frac{(2x+9)\left(\frac{1}{x} + 28x^6\right) - (\ln x + 4x^7)(2)}{(2x+9)^2}}$$

(g) $f(x) = \frac{(2 \cdot 3^{-5x})^4}{27^{x+2}}$ simplify!

$$f(x) = \frac{2^4 \cdot (3^{-5x})^4}{(3^3)^{x+2}} = \frac{16 \cdot 3^{-20x}}{3^{3x+6}} = 16 \cdot 3^{-20x-(3x+6)} = 16 \cdot 3^{-23x-6}$$

$$f(x) = 16 \cdot 3^{-23x-6}$$

$$\boxed{f'(x) = 16(\ln 3) \cdot \left(3^{-23x-6}\right)(-23)}$$

21. Solve each of the following for x or b as indicated.

(a) $\log_b 64 = \frac{3}{2}$

$$\left(b^{3/2}\right)^{\frac{2}{3}} = (64)^{\frac{2}{3}}$$

$$\boxed{b = 16}$$

(b) $\log_9 x = -3$

$$9^{-3} = x \longrightarrow \boxed{x = \frac{1}{729}}$$

(c) $5^{x^2-4x} = \frac{1}{25^x}$

$$5^{x^2-4x} = (5^2)^{-x}$$

$$5^{x^2-4x} = 5^{-2x}$$

$$x^2 - 4x = -2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$\boxed{x = 0, \ x = 2}$$

(d) $3^x \cdot 8x + 3^{x+1} = 0$

$$3^x \cdot 8x + 3 \cdot 3^x = 0$$

$$3^x(8x+3) = 0$$

$3^x = 0$ no solution

$8x + 3 = 0$

$8x = -3$

$$\boxed{x = -3/8}$$

(e) $\log_2(x+2) + \log_2 x = 2\log_2 3 + \log_2 10 - \log_2 6$

$$\log_2[(x+2)x] = \log_2 3^2 + \log_2 \frac{10}{6}$$

$$\log_2(x^2+2x) = \log_2\left(9 \cdot \frac{10}{6}\right)$$

$$\log_2(x^2+2x) = \log_2(15)$$

$$x^2 + 2x = 15$$

$$x^2 + 2x - 15 = 0$$

$$(x+5)(x-3) = 0$$

$x = -5$ ↑ not in domain!

$\boxed{x = 3 \text{ only}}$

22. In 1965, Bob invested \$300 in a savings account paying 4.2% per year compounded continuously.

(a) What was the $\boxed{\text{average rate of change}}$ of this account's value from 1972 to 1975?

$$\frac{A(b)-A(a)}{b-a} \qquad a=7 \quad b=10$$

$$\underbrace{A=300e^{0.042t}}_{Y_1} \qquad \frac{Y_1(10)-Y_1(7)\text{ dollars}}{10-7\text{ yr}} = \boxed{\$18.02\text{ per year}}$$

(b) What was the $\boxed{\text{average value}}$ (i.e., average balance) of this account from 1972 to 1975?

$$\frac{1}{b-a}\int_a^b A(t)\,dt$$

$$=\frac{1}{10-7}\int_7^{10} 300e^{0.042t}\,dt$$

$$=\frac{1}{3}\text{fnInt}(Y_1, X, 7, 10) = \frac{1}{3}(1286.983228) = \boxed{\$428.99}$$

23. Find the intervals where $f(x) = 3e^{2x} - 8e^x$ is concave upward and concave downward, and find the coordinates of any inflection points.

$f''(x)$: $- - - \mid + + +$ at $\ln\frac{2}{3}$

≈ -0.4055

infl. pt here

Test#	$f''(x)$
-1	-1.3190
0	4

$$f'(x) = 3e^{2x}(2) - 8e^x$$

$$f'(x) = 6e^{2x} - 8e^x$$

$$f''(x) = 6e^{2x}(2) - 8e^x$$

$$f''(x) = 12e^{2x} - 8e^x = 0$$

$$4e^x(3e^x - 2) = 0$$

$4e^x = 0$ no soln

$3e^x - 2 = 0$

$3e^x = 2$

$e^x = \frac{2}{3}$

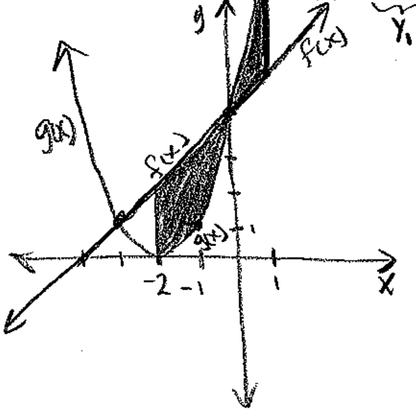
$\ln e^x = \ln\frac{2}{3}$

$x = \ln\frac{2}{3}$ in domain ✓

$\underbrace{f}_{\text{original function}}(\ln\frac{2}{3}) = -4$

$f(x)$ is concave upward on $(\ln\frac{2}{3}, \infty)$ and concave downward on $(-\infty, \ln\frac{2}{3})$. $f(x)$ has an inflection point: $(\ln\frac{2}{3}, -4)$.

24. Find the area between $f(x) = \underbrace{x+4}_{Y_1}$ and $g(x) = \underbrace{(x+2)^2}_{Y_2}$ on the interval $[-2, 1]$.



$$\text{Area} = \int_{-2}^{0}(f(x)-g(x))dx + \int_0^1 (g(x)-f(x))dx$$

$$= \text{fnInt}(Y_1 - Y_2, X, -2, 0) + \text{fnInt}(Y_2 - Y_1, X, 0, 1)$$

$$= \frac{10}{3} + \frac{11}{6}$$

$$= \boxed{\frac{31}{6}}$$

25. The management of Bob's Auto Shop has determined the price-demand equation for its air filters to be $2x + 100p = 2000$ where x is the number of filters sold when the unit price is $\$p$.

(a) Find the elasticity of demand $E(p)$.

$2x = 2000 - 100p$

$f(p) = x = 1000 - 50p$

$f'(p) = -50$

$$E(p) = \frac{-pf'(p)}{f(p)} = \frac{-p(-50)}{1000-50p} = \frac{50p}{50(20-p)}$$

$$\boxed{E(p) = \frac{p}{20-p}}$$

(b) Find and interpret $E(9)$.

$$E(9) = \frac{9}{20-9} = \boxed{\frac{9}{11}} < 1$$

Demand is inelastic when the unit price is \$9. This means that a small change in price will produce a smaller change in demand.

(c) If the current price of \$9 per air filter is increased by 3%, what would be the approximate change in demand?

$$E(9) * 3\% = \frac{9}{11} * 3\% = 2.4545\%$$

Demand will decrease by approximately 2.4545%.

(d) If the current price per air filter is \$13, should Bob increase or decrease this price to produce an increase in revenue?

$$E(13) = \frac{13}{20-13} = \frac{13}{7} > 1$$ demand is elastic

Decrease price to produce an increase in revenue since demand is elastic.

26. A company's marginal profit can be modeled by $P'(x) = 18\sqrt{x+5}$ dollars per item, where x is the number of items produced and sold. If this company's profit is \$11,475 when 95 items are produced and sold, find $P(x)$.

$$P(x) = \int 18\sqrt{x+5}\,dx$$ $u = x+5$, $du = dx$

$$= 18\int u^{1/2}\,du$$

$$= 18\left(\frac{2}{3}u^{3/2}\right) + C$$

$$P(x) = 12(x+5)^{3/2} + C$$

$$P(95) = 12(95+5)^{3/2} + C = 11475$$

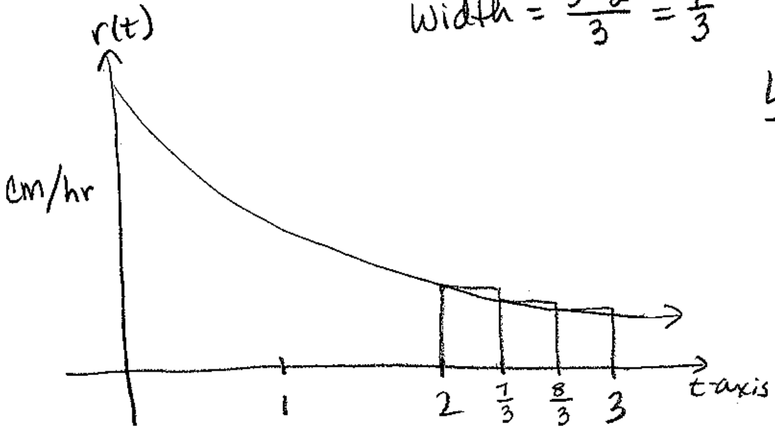
$$C = -525$$

$$\boxed{P(x) = 12(x+5)^{\frac{3}{2}} - 525 \text{ dollars}}$$

27. On a particular day during monsoon season, rain was falling at a rate given by $r(t) = \frac{5}{t+2}$ cm per hour, where t is the number of hours since 3am, $0 \le t \le 21$.

(a) Approximate the area under $r(t)$ from $t = 2$ to $t = 3$ using a left sum with 3 rectangles. What does this area represent?

Width $= \frac{3-2}{3} = \frac{1}{3}$



cm/hr * hrs = cm

Left Endpt	Height	Width	Area
2	$r(2) = \frac{5}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$\frac{7}{3}$	$r(\frac{7}{3}) = \frac{15}{13}$	$\frac{1}{3}$	$\frac{5}{13}$
$\frac{8}{3}$	$r(\frac{8}{3}) = \frac{15}{14}$	$\frac{1}{3}$	$+\ \frac{5}{14}$
			$\boxed{\frac{1265}{1092}}$ cm

≈ 1.1584 cm

Short method

Area $= \frac{1}{3}\left(r(2) + r(\frac{7}{3}) + r(\frac{8}{3})\right) = \boxed{\frac{1265}{1092}\text{ cm}}$

This area represents the total amount of rainfall from 5am to 6am on this day (1.1584cm).

(b) Find the exact amount of rain that fell from 8am to 2pm this day.

8am → $t = 5$
2pm → $t = 11$

$$\int_5^{11} \frac{5}{t+2}\,dt$$

$$= \text{fnInt}\left(\frac{5}{t+2}, t, 5, 11\right) = \boxed{3.0952\text{ cm}}$$

28. Find the intervals where $f(x) = \frac{8}{x} + x$ is increasing and decreasing, and find the coordinates of any local extrema.

$f(x) = 8x^{-1} + x$

$f'(x) = -8x^{-2} + 1$

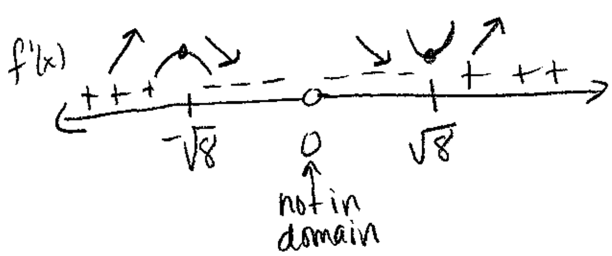
$= \frac{-8 + x^2}{x^2} = 0$

$-8 + x^2 = 0$

$x^2 = 8$

$x = \pm\sqrt{8}$ In domain, so critical numbers

Also, $f'(x)$ DNE for $x = 0$.



Test #	$f'(x)$
-3	1/9
-1	-7
1	-7
3	$\frac{1}{9}$

$f(-\sqrt{8}) = \frac{8}{-\sqrt{8}} - \sqrt{8} = -2\sqrt{8}$

$f(\sqrt{8}) = \frac{8}{\sqrt{8}} + \sqrt{8} = 2\sqrt{8}$

$f(x)$ is increasing on $(-\infty, -\sqrt{8})$ and $(\sqrt{8}, \infty)$. $f(x)$ is decreasing on $(-\sqrt{8}, 0)$ and $(0, \sqrt{8})$. $f(x)$ has a local maximum at the point $(-\sqrt{8}, -2\sqrt{8})$ and a local minimum at $(\sqrt{8}, 2\sqrt{8})$.

29. Compute each of the following.

(a) $\int \left(4e^x - 7x^3 + 4x^{-1} - e^2\right) dx$

$= 4e^x - 7\left(\frac{x^4}{4}\right) + 4\ln|x| - e^2 x + C$

$= \boxed{4e^x - \frac{7}{4}x^4 + 4\ln|x| - e^2 x + C}$

(b) $\int 4t^3\sqrt{t^4 - 7}\,dt$

$= \int \sqrt{t^4-7}\,(4t^3)\,dt$

$u = t^4 - 7$

$du = 4t^3\,dt$

$= \int \sqrt{u}\,du$

$= \frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{3}(t^4-7)^{\frac{3}{2}} + C}$

(c) $\int \frac{x^6}{4x^7 - 5}dx$

$u = 4x^7 - 5$

$du = 28x^6\,dx$

$= \frac{1}{28}\int \frac{28x^6}{4x^7-5}dx$

$= \frac{1}{28}\int \frac{1}{u}du$

$= \frac{1}{28}\ln|u| + C = \boxed{\frac{1}{28}\ln|4x^7-5| + C}$

(d) $\int (4x+2)e^{x^2+x}dx$

$= \int e^{x^2+x}(4x+2)\,dx$

$u = x^2 + x$

$du = (2x+1)\,dx$

$2du = (4x+2)\,dx$

$= \int e^u(2)\,du$

$= 2\int e^u\,du$

$= 2e^u + C = \boxed{2e^{x^2+x} + C}$

(e) $\int \frac{2}{x\sqrt{5+\ln x}}dx$

$u = 5+\ln x$

$du = \frac{1}{x}dx$

$= 2\int \frac{1}{\sqrt{5+\ln x}}\left(\frac{1}{x}\right)dx$

$= 2\int \frac{1}{\sqrt{u}}du$

$= 2\int u^{-\frac{1}{2}}du$

$= 2(2u^{1/2}) + C = \boxed{4(5+\ln x)^{\frac{1}{2}} + C}$

30. Let $f(x,y) = \frac{7x^2y^4}{x^3y-7y}$.

(a) Find $f(2,10)$.

$$f(2,10) = \frac{7(2)^2(10)^4}{(2^3)(10)-7(10)} = \boxed{28,000}$$

(b) Find the first-order partial derivatives of $f(x,y)$.

$$f_x(x,y) = \frac{(x^3y-7y)(14xy^4) - (7x^2y^4)(3x^2y)}{(x^3y-7y)^2}$$

$$f_y(x,y) = \frac{(x^3y-7y)(28x^2y^3) - (7x^2y^4)(x^3-7)}{(x^3y-7y)^2}$$

31. Let $Q(x)$ be an antiderivative of $q(x)$. Find $Q(3)$ if $\int_3^7 q(x)dx = -9$ and $Q(7) = 5$.

$$\int_3^7 q(x)dx = Q(x)\Big|_3^7 = Q(7) - Q(3) = -9$$

$$5 - Q(3) = -9$$

$$-Q(3) = -14$$

$$\boxed{Q(3) = 14}$$

32. Mariska is selling lemonade. When she charges \$5 per glass, she can sell 36 glasses of lemonade. For every \$0.10 that she lowers the price, an additional 4 glasses of lemonade will be sold. What price should Mariska charge per glass to maximize revenue?

Let x be the number of \$0.10 price reductions.

$\boxed{\text{Maximize } R(x) = (5 - 0.10x)(36 + 4x) \text{ where } x \leq 50.}$

Domain

price $= 5 - 0.1x \geq 0$

$-0.1x \geq -5$

$x \leq 50$

$R(x) = 180 + 20x - 3.6x - 0.4x^2$

$R(x) = 180 + 16.4x - 0.4x^2$

Method 1: Algebra

$R(x)$ is a parabola opening downward, so a maximum occurs at the vertex:

$x = \frac{-b}{2a} = \frac{-16.4}{2(-0.4)} = 20.5$

price $= 5 - 0.1(20.5) = \boxed{\$2.95 \text{ per glass}}$

Method 2: Calculus

$R'(x) = 16.4 - 0.8x = 0$

$-0.8x = -16.4$

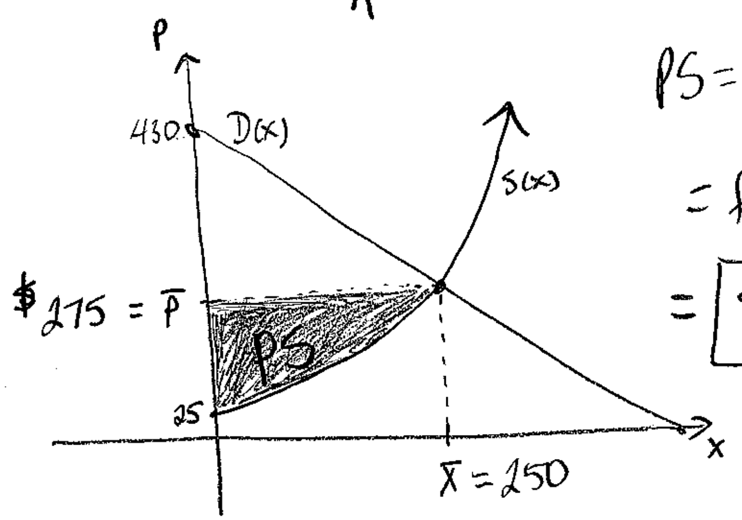
$x = 20.5$

$R''(x) = -0.8 < 0$ max

price $= 5 - 0.1(20.5)$

$= \boxed{\$2.95 \text{ per glass}}$

33. Find the producers' surplus at equilibrium price level for a product whose price-demand equation is given by $p = D(x) = \underbrace{430 - 0.62x}_{Y_1}$ dollars per item and price-supply equation is given by $p = S(x) = \underbrace{0.004x^2 + 25}_{Y_2}$ dollars per item.



$$PS = \int_0^{250} (275 - S(x))dx$$

$$= \text{fnInt}(275 - Y_2, X, 0, 250)$$

$$= \boxed{\$41{,}666.67 \text{ gained by producers}}$$

Graph Y_1, Y_2, calc. Intersect $(250, 275)$ = equilibrium pt.

34. The productivity of an automobile parts manufacturing company is given by $f(x,y) = 30x^{0.35}y^{0.65}$ units, where x and y represent the number of units of labor and capital utilized, respectively.

(a) Find $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = 30(0.35)x^{-0.65}y^{0.65}$$

$$\boxed{f_x(x,y) = 10.5x^{-0.65}y^{0.65}}$$

$$f_y(x,y) = 30x^{0.35}(0.65)y^{-0.35}$$

$$\boxed{f_y(x,y) = 19.5x^{0.35}y^{-0.35}}$$

(b) Find the $\boxed{\text{marginal productivity of labor}}$ when 70 units of labor and 50 units of capital are utilized.

$$f_x(70,50) = 10.5(70)^{-0.65}(50)^{0.65} = \boxed{8.4374 \text{ units per unit of labor}}$$

(c) Find the $\boxed{\text{marginal productivity of capital}}$ when 70 units of labor and 50 units of capital are utilized.

$$f_y(70,50) = 19(70)^{0.35}(50)^{-0.35} = \boxed{21.3746 \text{ units per unit of capital}}$$

(d) For the greatest increase in productivity, should this company increase the use of labor or capital (assuming they are currently using 70 units of labor and 50 units of capital)?

increase the use of capital since the marginal productivity of capital is larger.

35. Find all second-order partial derivatives of $f(x,y) = 2xe^{xy}$.

$$f_x(x,y) = 2e^{xy} + 2xe^{xy}(y)$$

$$f_x(x,y) = 2e^{xy}(1+xy)$$

$$\boxed{f_{xx}(x,y) = 2e^{xy}(y)(1+xy) + 2e^{xy}(y)}$$

$$\boxed{f_{xy}(x,y) = 2e^{xy}(x)(1+xy) + 2e^{xy}(x)}$$

$$= 2xe^{xy}[(1+xy)+1]$$

$$f_{xy}(x,y) = 2xe^{xy}(xy+2) \quad \text{(Simplified, we see that } f_{xy} = f_{yx}\text{)}.$$

$$f_y(x,y) = 2xe^{xy}(x)$$

$$f_y(x,y) = 2x^2e^{xy}$$

$$\boxed{f_{yy}(x,y) = 2x^2e^{xy}(x)}$$

$$\boxed{f_{yx}(x,y) = 4xe^{xy} + 2x^2e^{xy}(y)}$$

$$f_{yx}(x,y) = 2xe^{xy}(2+xy)$$