Math 166 - Week in Review #11

Section 9.4 - Game Theory and Strictly Determined Games

- **Zero-sum Game** - a game in which the payoff to one party results in an equal loss to the other.
- The entries in a payoff matrix represent the earnings of the row player.
- **Maximin Strategy**
  1. For each **row** of the payoff matrix, find the smallest entry in that row and underline it.
  2. Find the largest underlined number and note the row that it is in. This row gives the row player's "best" move.
- **Minimax Strategy**
  1. For each **column** of the payoff matrix, find the largest entry in that column and $\boxed{\text{circle}}$ it.
  2. Find the smallest circled number and note the column that it is in. This column gives the column player's "best" move.
- Strictly Determined Game - A strictly determined game has the following properties:
  1. There is an entry in the payoff matrix that is simultaneously the smallest entry in its row and the largest entry in its column (i.e., there is one entry that is both underlined and circles at the same time). This entry is called the **saddle point** for the game.
  2. The optimal strategy for the row player is precisely the maximin strategy and is the row containing the saddle point. The optimal strategy for the column player is the minimax strategy and is the column containing the saddle point.
- The saddle point of a strictly determined game is also referred to as the **value of the game**.

Section 9.5 - Game Theory and Strictly Determined Games

- The **expected value of a game** measures the average payoff to the row player when both players adopt a particular set of mixed strategies.
- The expected value of a game, E, can be found by computing the matrix product

$$E = PAQ$$

  where P is the row player's mixed strategy, A is the payoff matrix for the game, and Q is the column player's mixed strategy.

- Optimal Strategies for Nonstrictly Determined Games

  Let $A = \begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$ be the payoff matrix for a nonstrictly determined game. Then the optimal mixed strategy for the row player is given by $P = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, where

$$p_1 = \frac{d-c}{a+d-b-c} \text{ and } p_2 = 1 - p_1$$

  and the optimal mixed strategy for the column player is given by $Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, where

$$q_1 = \frac{d-b}{a+d-b-c} \text{ and } q_2 = 1 - q_1$$

  Also, the expected value of the game, $E = PAQ$, can be computed as

$$E = \frac{ad-bc}{a+d-b-c}$$

1. Each of the following matrices represents the payoff in a two-person zero-sum game. Determine the maximin and minimax strategies for each player. If there is a saddle point, find it and determine the value of the game.

(a) $\begin{array}{cc} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} -2 \\ 0 \end{array} \\ \begin{array}{cc} 2 & 1 \end{array} & \end{array}$ $\leftarrow R_2$ = maximin strategy of the row player.

C_2 = minimax strategy for the column player.

No saddle point.

(b) $\begin{array}{cc} \begin{bmatrix} 5 & -8 \\ 3 & 1 \end{bmatrix} & \begin{array}{c} -8 \\ 1 \end{array} \\ \begin{array}{cc} 5 & 1 \end{array} & \end{array}$ $\leftarrow R_2$ = maximin strategy

C_2 = minimax strategy

Is a saddle point at a_{22} and the value of the game 1.

(c) $\begin{array}{cc} \begin{bmatrix} -1 & 5 \\ 1 & 4 \\ 0 & -1 \end{bmatrix} & \begin{array}{c} -1 \\ 1 \\ -1 \end{array} \\ \begin{array}{cc} 1 & 5 \end{array} & \end{array}$ $\leftarrow R_2$ = maximin strategy

C_1 = minimax strategy

Saddle point at a_{21} and the value of the game is 1.

(d) $\begin{array}{cc} \begin{bmatrix} 4 & 3 & -1 \\ 2 & 5 & -5 \\ 1 & 0 & 2 \end{bmatrix} & \begin{array}{c} -1 \\ -5 \\ 0 \end{array} \\ \begin{array}{ccc} 4 & 5 & 2 \end{array} & \end{array}$ $\leftarrow R_3$ = maximin strategy

C_3 = minimax strategy

No saddle point.

2. Robert and Camille play a game in which each casts a fair four-sided die at the same time. If the sum of the numbers landing up is less than 4, then Camille pays Robert \$10. If the sum of the numbers landing up is greater than 4, then Robert pays Camille \$2. However, if the sum is exactly 4, then no payoff is made to either player.

(a) Find the payoff matrix for this two-player zero-sum game.

Camille

$$\text{Robert}\quad \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 10 & 10 & 0 & -2 \\ 10 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{bmatrix} \end{array}$$

(b) Find the maximin and minimax strategies for Robert and Camille. Is this game strictly determined?

Camille

$$\text{Robert}\quad \begin{array}{c|cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 10 & 10 & 0 & -2 \\ 10 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{bmatrix} & \left.\begin{array}{c} -2 \\ -2 \\ -2 \\ -2 \end{array}\right\} \text{all equal, so any row} \\ & \begin{array}{cccc} 10 & 10 & 0 & -2 \end{array} & \end{array}$$

$\uparrow$ C4

Maximin Strategy: Row 1, 2, 3, or 4.

Minimax Strategy: Column 4

Since the payoff matrix has a saddle point (4 locations, actually), the game is strictly determined.

(Value of game: −2)
(game favors Camille)

Title: Apr 30-6:41 PM (3 of 10)

3. In the game of "paper, rock, scissors," two players simultaneously show a hand signal representing one of paper, rock, and scissors. In this game, "paper" beats "rock" (since paper can smother the rock), "rock" beats "scissors" (since a rock can crush scissors), and "scissors" beats "paper" (since scissors can cut paper). Suppose two people play this game and each time, the loser must pay the winner $1.

(a) Write the payoff matrix for this game.

Column

Row

$$\begin{array}{ccc} & \begin{array}{ccc} \text{paper} & \text{rock} & \text{scissors} \end{array} & \\ \begin{array}{c} \text{paper} \\ \text{rock} \\ \text{scissors} \end{array} & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} & \begin{array}{c} -1 \\ -1 \\ -1 \end{array} \\ & \begin{array}{ccc} 1 & 1 & 1 \end{array} & \end{array}$$

(b) Determine the maximin and minimax strategies for the row and column players. Is this game strictly determined?

Maximin Strategy: Play Row 1, 2, or 3

Minimax Strategy: Play Column 1, 2, or 3

No saddle point so <u>not</u> strictly determined.

4. TV stations R and C each have a quiz show and sitcom to schedule for their 1pm and 2pm time slots. If they both schedule their quiz shows at 1pm, then station R will take \$3,000 in advertising revenue away from station C. If they both schedule their quiz shows at 2pm, then station C will take \$2,000 in advertising revenue from R. If they choose different hours for the quiz show, then R will take \$5,000 in advertising from C by scheduling it at 2pm and \$2,000 by scheduling it at 1pm.

(a) Give the payoff matrix.

Station C

$$\text{Station } R \quad \begin{array}{c} \\ 1\text{pm} \\ 2\text{pm} \end{array} \begin{array}{c} \begin{array}{cc} 1\text{pm} & 2\text{pm} \end{array} \\ \begin{bmatrix} 3000 & 2000 \\ 5000 & -2000 \end{bmatrix} \end{array} \begin{array}{l} \\ 2000 \leftarrow R_1 \\ -2000 \end{array}$$

$$\begin{array}{cc} 5000 & 2000 \leftarrow C_2 \end{array}$$

(b) Is this game strictly determined? If yes, give the optimal strategies for R and C and state the value of the game.

Yes since there is a saddle point.

The optimal strategy for Station R is to schedule the quiz show at 1pm and the optimal strategy for Station C is to schedule it at 2pm.

5. A farmer is trying to decide whether or not to expand his production of corn to a higher level. He has determined that if he expands his corn production and the growing season is drier than normal, he will have a profit of \$3,500. If he expands production and the growing season has an average amount of rainfall, then he will make a profit of \$6,000. If he expands production and the growing season is wetter than usual, he will make a profit of \$7,500. If he does not expand production and the growing season is drier than normal, average, or wetter than average, he will make profits of \$2,500, \$3,500, and \$4,000.

(a) Represent this information in the form of a payoff matrix.

$$
\begin{array}{cc}
 & \text{weather} \\
 & \begin{array}{ccc} \text{dry} & \text{normal} & \text{wet} \end{array} \\
\text{farmer} \begin{array}{r} \text{expand} \\ \text{don't expand} \end{array} & \begin{bmatrix} \underline{3500} & 6000 & 7500 \\ \underline{2500} & 3500 & 4000 \end{bmatrix} \begin{array}{l} 3500 \leftarrow \\ 2500 \end{array}
\end{array}
$$

(b) Assuming that the weather for the coming growing season is unpredictable, determine whether or not the farmer should expand his corn production.

Using the maximin strategy, the farmer determines that he should expand production.

6. Refer to the game in #3. Find the expected payoff to the row player if

(a) the row player decides to show paper, rock, and scissors the same proportion of the time each and the column player decides to show paper, rock, and scissors 30%, 40%, and 30% of the time each.

$E = PAQ$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad Q = \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix}$$

$E = PAQ = 0$ The row player's expected avg winnings per play is \$0.

(b) Who does the game favor in part (a)?

Neither player – It is a fair game with these mixed strategies.

(c) the row player decides to show rock 50% of the time and the other two options an equal amount of time and the column player decides to show rock 65% of the time and paper 25% of the time.

$E = PAQ$

$$P = \begin{bmatrix} .25 & .5 & .25 \end{bmatrix}, \quad Q = \begin{bmatrix} .25 \\ .65 \\ .1 \end{bmatrix}$$

$E = PAQ = -0.0375$

The row player can expect to lose 3.75¢ per play on average with these mixed strategies.

(d) Who does the game favor in part (c)?

Column player.

7. Rob and Cathy are running for student body president. If both candidates campaign only to upperclassmen, then Rob is expected to get 58% of the votes. If both candidates campaign only to underclassmen, then Cathy is expected to get 62% of the votes. If Rob campaigns only to upperclassmen while Cathy campaigns only to underclassmen, Rob is expected to get 70% of the vote. If the reverse of this occurs, Cathy is expected to get 40% of the vote.

(a) Construct the payoff matrix for this game and show that it is not strictly determined.

$$\begin{array}{c|cc} & \text{Cathy upper} & \text{under} \\ \hline \text{Rob upper} & .58 & .70 \\ \text{under} & .60 & .38 \end{array}$$

$.58 \leftarrow R_1$

$.38$

C_1

no saddle pt, so not strictly determined.

(b) Find the optimal strategy for both Rob and Cathy.

Payoff Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Rob: $P = [p_1 \quad p_2]$

$$p_1 = \frac{d-c}{a+d-b-c} = \frac{.38-.6}{.58+.38-.7-.6}$$

$$p_1 = \frac{11}{17} \qquad p_2 = 1 - \frac{11}{17} = \frac{6}{17}$$

Rob's optimal mixed strategy is $P = \left[\frac{11}{17} \quad \frac{6}{17}\right]$.

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad q_1 = \frac{d-b}{a+d-b-c} = \frac{.38-.7}{.58+.38-.7-.6} = \frac{16}{17}$$

$$q_2 = 1 - \frac{16}{17} = \frac{1}{17}$$

$Q = \begin{bmatrix} 16/17 \\ 1/17 \end{bmatrix}$ = Cathy's optimal mixed strategy.

(c) If each candidate employs the optimal strategy, who can be expected to win?

$E = PAQ$ or you can use the formula:

$$E = \frac{ad-bc}{a+d-b-c} = \frac{(.58)(.38)-(.7)(.6)}{.58+.38-.7-.6}$$

$$= 0.5871 \quad \text{(to 4 decimal places)}$$

Since Rob is expected to get 58.71% of the vote, he is expected to win.

8. You have \$50,000 to invest in stocks and bonds. Your financial advisor has informed you that the annual return on your investment depends on the state of the economy. In an expanding economy, the investment in stocks will increase by 15% and the investment in bonds will increase by 8%. In economic recession, the investment in stocks will decrease by 3% and the investment in bonds will increase by 10%.

(a) Write the payoff matrix for this game.

Economy

$$\text{You:}\quad \begin{array}{c c} & \text{expand} \quad \text{recession} \\ \begin{array}{c}\text{stocks}\\ \text{bonds}\end{array} & \begin{bmatrix} .15 & -.03 \\ .08 & .10 \end{bmatrix} \end{array}$$

(b) What is your optimal investment strategy?

$$P = [p_1 \quad p_2]$$

$$p_1 = \frac{d-c}{a+d-b-c} = \frac{.1-.08}{.15+.1+.03-.08} = .1$$

Invest 10% in stocks

$$p_2 = 1 - .1 = .9$$ ← Invest 90% in bonds

$.1(50,000) = \$5,000$ should be invested in stocks

$.9(50000) = \$45000$ invested in bonds

(c) What profit can you expect by using your optimal investment strategy?

$E = PAQ$ or

$$E = \frac{ad-bc}{a+d-b-c} = \frac{(.15)(.1)-(-.03)(.08)}{.15+.1+.03-.08}$$

$$= 0.087$$

Expected profit of 8.7% which means

$.087 * 50000 = \boxed{\$4,350 \text{ expected profit}}$