Section 2.1 - Systems of Linear Equations

- When a system of linear equations has only two variables, each equation represents a line and "solving the system" means finding all points the lines have in common.
- For any system of linear equations (with finitely many variables), there are only 3 possibilities for the solution: (1) a unique solution, (2) infinitely many solutions, or (3) no solution.
- If a system of equations has infinitely many solutions, you MUST give the parametric solution for the system.

Section 2.2 - Systems of Linear Equations: Unique Solutions

- Row-Reduced Form of a Matrix
  1. Each row consisting entirely of zeros lies below any other row having nonzero entries.
  2. The first nonzero entry in each row is 1 (called a **leading 1**).
  3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
  4. If a column contains a leading 1, then the other entries in that column are zeros.

  **NOTE:** We only consider the coefficient side (the left side) of an augmented matrix when determining whether the matrix is in row reduced form.

  Gauss-Jordan Row Operations
  1. Interchange any two rows ($R_i \leftrightarrow R_j$).
  2. Replace any row by a nonzero constant multiple of itself ($cR_i$).
  3. Replace any row by the sum of that row and a constant multiple of any other row ($R_i + aR_j$).

- Gauss-Jordan Elimination Method
  1. Write the given linear system in augmented matrix form.
  2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry of the first row (the $a_{11}$ entry) is nonzero. The $a_{11}$ entry is the first "pivot element."
  3. Pivot about $a_{11}$, i.e., transform the $a_{11}$ entry to a 1 and then transform the other elements in the first column into zeros using the 3 row operations.
  4. Now interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row (the $a_{22}$ entry) is nonzero. This is the second pivot element.
  5. Pivot about $a_{22}$.
  6. Continue in a similar manner until the the left side of the augmented matrix is in row-reduced form.

- *Pivoting* on an entry in a matrix means transform the pivot element into a 1 and then transform all other entries in the same column into 0's.

1. Solve the system of equations

$$2\left(x-\frac{3}{2}y = -2\right) \longrightarrow 2x-3y=-4$$
$$6\left(\frac{2}{3}x-\frac{5}{6}y = \frac{7}{3}\right) \longrightarrow 4x-5y=14$$

$$-2(2x-3y=-4) \rightarrow -4x+6y=8$$
$$4x-5y=14 \rightarrow +\ 4x-5y=14$$
$$\boxed{y=22}$$

$$2x-3(22)=-4$$
$$2x=62$$
$$\boxed{x=31}$$

$$\boxed{(x,y)=\boxed{(31,22)}\ \text{unique solution}}$$

2. Solve the system of equations $\begin{aligned} 2(3x-6y &= 18) \\ 3(-2x+4y &= -12) \end{aligned}$

$$\begin{aligned} 6x - 12y &= 36 \\ + \quad -6x + 12y &= -36 \\ \hline 0 &= 0 \end{aligned}$$

TRUE!

Infinitely Many Solutions

We must give the parametric solution.

Step1: Let $x = t$, where $t$ is any real #.

(Now solve for y.)

$$-2x + 4y = -12$$

$$4y = 2x - 12$$

$$y = \frac{1}{2}x - 3$$

(Now substitute t for x.)

$$y = \frac{1}{2}t - 3$$

$(x, y) = \boxed{\left(t, \frac{1}{2}t - 3\right)}$ the parametric soln.

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3. (a) Find the value of $k$ so that the given system has no solution.

$$7x - 5y = -3$$
$$3x + ky = 15$$

- parallel lines
- same slope
- different y-int's

$7x - 5y = -3$

$-5y = -7x - 3$

1st line $y = \frac{7}{5}x + \frac{3}{5}$

$m = \frac{7}{5}$

$3x + ky = 15$

$ky = -3x + 15$ 2nd line

$y = -\frac{3}{k}x + \frac{15}{k}$

$m = -\frac{3}{k}$

set equal

$\frac{7}{5} = -\frac{3}{k}$

$7k = -15$

$k = -\frac{15}{7}$

Now check that $k = -\frac{15}{7}$ makes the two y-intercepts different.

first line: $b = \frac{3}{5}$

2nd line: $b = \frac{15}{k} = \frac{15}{-15/7} = \frac{\cancel{15}}{1} \cdot -\frac{7}{\cancel{15}} = -7$

different from 1st line.

Ans: $\boxed{k = -\frac{15}{7}}$

(b) Is it possible to find a value of $k$ so that the system has infinitely many solutions? Explain.

No. The $k$ that forces the slopes of the two lines to be equal ($k = -\frac{15}{7}$), makes the y-intercepts be different #'s. Thus, the 2 lines are not the same.

(c) For what value(s) of $k$ will the system have a unique solution?

Any value of $k$ other than $-\frac{15}{7}$ will result in 2 lines with different slopes, and therefore a unique solution (one pt. of intersection).

$\boxed{k \neq -\frac{15}{7}}$

**For the next 3 exercises, set up the system of equations but do not solve.**

4. (#49, pg. 74 of *Finite Mathematics* by Lial, et. al.) The U-Drive Rent-A-Truck Co. plans to spend \$6 million on 200 new vehicles. Each van will cost \$20,000, each small truck \$30,000, and each large truck \$50,000. Past experience shows that they need twice as many vans as small trucks. How many of each kind of vehicle can they buy?

1) Define the variables.

Let $x$ = the number of vans they can buy.
Let $y$ = the number of small trucks they can buy.
Let $z$ = the number of large trucks they can buy.

2) Set up a system of equations.

1) $x + y + z = 200$

2) $20000x + 30000y + 50000z = 6000000$

Reword the "as many as" statement:

The number of vans is two times the number of small trucks

3) $x = 2y$

System:

$$x + y + z = 200$$
$$20000x + 30000y + 50000z = 6,000,000$$
$$x = 2y$$

5. A cashier has a total of 96 bills in his register in one-, five-, and ten-dollar denominations. If he has three times as many fives as ones, and if the number of ones and fives combined is half of the number of tens he has, how many bills of each denomination does he have in his register?

Let $x$ = the number of ones in the register.
Let $y$ = the number of fives in the register.
Let $z$ = the number of tens in the register.

$$x + y + z = 96$$

The # of 5's is 3 times the # of 1's

$$y = 3x$$

$$x + y = \frac{1}{2}z$$

System:
$$\boxed{\begin{aligned} x + y + z &= 96 \\ y &= 3x \\ x + y &= \tfrac{1}{2}z \end{aligned}}$$

6. Random, Inc. makes picture collages in three sizes. A small collage requires 30 minutes of cutting time and 36 minutes of pasting time. A medium collage requires 60 minutes of cutting time and 54 minutes for pasting. A large collage requires 90 minutes for cutting and 72 minutes for pasting. There are 380 labor hours available for cutting and 330 labor hours available for pasting each week. If the company wants to run at full capacity and wants to make twice as many small collages as medium collages, how many collages of each size should be made each week?

Let $x$ = the number of small collages.
Let $y$ = the number of medium collages.
Let $z$ = the number of large collages.

| | Small | Medium | Large | Time Available |
|---|---|---|---|---|
| Cutting time | 30min | 60min | 90min | 380hrs * 60 = 22,800 min |
| pasting time | 36min | 54min | 72min | 330hrs * 60 = 19,800 min |

$$30x + 60y + 90z = 22{,}800$$
$$36x + 54y + 72z = 19{,}800$$
$$x = 2y$$

transform the entry in row 1, column 1 into a 1 and then transform all other entries in column 1 into 0's.

7. Perform the first pivot in the Gauss-Jordan Elimination Method. Indicate all row operations used.

$$A = \left[\begin{array}{ccc|c} 3 & -6 & 0 & 9 \\ 7 & 4 & -8 & 6 \\ 2 & -3 & 1 & 9 \end{array}\right]$$

$$A \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 7 & 4 & -8 & 6 \\ 2 & -3 & 1 & 9 \end{array}\right] \xrightarrow{R_2 - 7R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 18 & -8 & -15 \\ 2 & -3 & 1 & 9 \end{array}\right]$$

$R_2 - 7R_1$

$[7\ 4\ -8\ 6] - 7[1\ -2\ 0\ 3]$

$= [7\ 4\ -8\ 6] + [-7\ 14\ 0\ -21]$

$= [0\ 18\ -8\ -15]$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 18 & -8 & -15 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$[2\ -3\ 1\ 9] - 2[1\ -2\ 0\ 3]$

$= [2\ -3\ 1\ 9] + [-2\ 4\ 0\ -6]$

$= [0\ 1\ 1\ 3]$

1st pivot is now complete.

8. For each of the following, determine if the matrix is in row reduced form. If it is, interpret its meaning as a solution to a system of equations. If the matrix is not in row reduced form, perform Gauss-Jordan row operations until it is and then interpret the solution.

(a) $\begin{array}{cc} x & y & \\ \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \end{array}$ Is in row-reduced form.

$1x + 0y = 5 \quad x = 5$

$0x + 1y = 3 \quad y = 3$

$0x + 0y = 0$

$0 = 0$ TRUE

$\boxed{\text{unique solution: } (5,3)}$

(b) $\left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 3 & -5 \\ 3 & 0 & -12 & 6 \end{array}\right]$ Not in row-reduced form because column 1 has a leading 1 but not all other entries in that column are 0.

$R_3 - 3R_1 \longrightarrow \boxed{\left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]}$ Is in row-red. form.

$[3 \;\; 0 \;\; -12 \;\; 6] - 3[1 \;\; 0 \;\; -4 \;\; 2]$

$= [3 \;\; 0 \;\; -12 \;\; 6] + [-3 \;\; 0 \;\; 12 \;\; -6]$

$= [0 \;\; 0 \;\; 0 \;\; 0]$

Copy:

$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \rightarrow x - 4z = 2 \\ \rightarrow y + 3z = -5 \\ \rightarrow 0 = 0 \text{ TRUE} \end{array}$

Infinitely many solutions

Let $z = t$ where $t$ is any real #.

(Now solve for x and y.)

$x = 4z + 2 \rightarrow x = 4t + 2$

$y = -3z - 5 \rightarrow y = -3t - 5$

$(x,y,z) = \boxed{(4t+2, -3t-5, t)}$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & -3 & -9 & 16 \end{array}\right]$ Not in row red. form

$R_3 + 3R_2$

work

$[0 \quad -3 \quad -9 \quad 16] + 3[0 \quad 1 \quad 3 \quad -5]$

$= [0 \quad -3 \quad -9 \quad 16] + [0 \quad 3 \quad 9 \quad -15]$

$= [0 \quad 0 \quad 0 \quad 1]$

New matrix is in row-red. form.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$\to x = 2$

$\to y + 3z = -5$

$\to 0x + 0y + 0z = 1$

$0 = 1$

FALSE!

$\boxed{\text{No Solution}}$

(d) $\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -7 \end{array}\right] \end{array}$ Is in row-red. form.

$1x + 3y + 0z = 5 \rightarrow$ $x + 3y = 5$

$0x + 0y + 1z = -7 \rightarrow$ $z = -7$

Problem continued on next page.

When finding the solution using an augmented matrix in row-reduced form,

1) <u>first</u> count the leading 1's.

- If the # of leading 1's equals the # of variables, there is a unique solution.

- If the # of leading 1's is less than the # of variables, then

  1) Check to see if there is no solution.

  2) If there is a solution, then it is parametric.

     - Pick the variable that does not have a leading 1 in its column to be the parameter.
     - Then solve for the other variables.

(d) continued:

Since $y$ doesn't have a leading 1 in its column,

Let $y = t$, where $t$ is any real #.

(Now solve for the other variables)

$z = -7$

$x = -3y + 5 \rightarrow x = \underline{-3t + 5}$

$(x, y, z) = \boxed{(-3t+5,\ t,\ -7)}$

parametric soln.