

## Math 141 - Week in Review #2

### Section 2.1 - Systems of Linear Equations

- When a system of linear equations has only two variables, each equation represents a line and “solving the system” means finding all points the lines have in common.
- For any system of linear equations (with finitely many variables), there are only 3 possibilities for the solution: (1) a unique solution, (2) infinitely many solutions, or (3) no solution.
- If a system of equations has infinitely many solutions, you **MUST** give the parametric solution for the system.

### Section 2.4 - Matrices

- The size of a matrix is always *number of rows*  $\times$  *number of columns*.
- $c_{ij}$  represents the entry of the matrix  $C$  in row  $i$  and column  $j$ .
- To add and subtract matrices, they must be the same size.
- When adding or subtracting matrices, add or subtract corresponding entries.
- A scalar product is computed by multiplying each entry of a matrix by a scalar (a number).
- Transpose - The rows of the matrix  $A$  become the columns of  $A^T$ .

### Section 2.5 - Multiplication of Matrices

- The matrix product  $AB$  can be computed only if the number of columns of  $A$  equals the number of rows of  $B$ .
- If  $C = AB$ , then  $c_{ij}$  is computed by multiplying the  $i^{\text{th}}$  row of  $A$  by the  $j^{\text{th}}$  column of  $B$ .
- Identity Matrix - Denoted by  $I_n$ , the identity matrix is the  $n \times n$  matrix with 1's down the main diagonal (from upper left corner to lower right corner) and 0's for all other entries.
- If  $A$  is  $m \times n$ , then  $AI_n = A$  and  $I_m A = A$ .

### Section 2.6 - The Inverse of a Square Matrix

- Only square matrices can have inverses, *but not all square matrices have inverses*.
- A square matrix that does not have an inverse is called a **singular matrix**.
- The inverse of  $A$ , denoted  $A^{-1}$ , is the square  $n \times n$  matrix such that  $AA^{-1} = A^{-1}A = I_n$ .
- Systems of equations can be represented as a matrix equation of the form  $AX = B$  where  $A$  is the coefficient matrix,  $X$  is a column vector containing the variables, and  $B$  is a column vector containing constants.
- If  $A$  has an inverse, the solution to the matrix equation is  $X = A^{-1}B$ .
- If  $A$  does not have an inverse (i.e., if  $A$  is singular), this does **NOT** imply the system has no solution. It simply means that you must use another method to solve the system.

1. Solve the system of equations

$$\begin{cases} 2(x - \frac{3}{2}y = -2) \\ 6(\frac{2}{3}x - \frac{5}{6}y = \frac{7}{3}) \end{cases} \rightarrow x = \frac{3}{2}y - 2$$

substitution

$$\frac{2}{3}(\frac{3}{2}y - 2) - \frac{5}{6}y = \frac{7}{3}$$

$$y - \frac{4}{3} - \frac{5}{6}y = \frac{7}{3} \quad +\frac{4}{3}$$

$$\frac{1}{6}y = \frac{11}{3} \quad \cdot \frac{6}{1}$$

$$y = \frac{66}{3} = 22$$

$$x = \frac{3}{2}(22) - 2$$

$$x = \frac{66}{2} - 2$$

$$x = 33 - 2 = 31$$

Can also clear denominators:

$$2x - 3y = -4$$

$$4x - 5y = 14$$

then use elimination method:

$$-2(2x - 3y = -4)$$

$$-4x + 6y = 8$$

$$+ 4x - 5y = 14$$

$$y = 22$$

$$2x - 3(22) = -4$$

$$2x = -4 + 66$$

$$2x = 62$$

$$x = 31$$

$$\boxed{\begin{matrix} x = 31 \\ y = 22 \end{matrix}}$$

or (31, 22)

2. Solve the system of equations

$$\begin{cases} 2(3x - 6y = 18) \\ 3(-2x + 4y = -12) \end{cases}$$

$$6x - 12y = 36$$

$$+ \quad -6x + 12y = -36$$

$0 = 0$  True! (same line)

Infinitely many solutions.

Let  $x = t$  where  $t$  is any real number,  
(solve for  $y$ )

$$3x - 6y = 18$$

$$-6y = -3x + 18$$

$$y = \frac{1}{2}x - 3$$

Parametric solution

$$x = t$$

$$y = \frac{1}{2}t - 3$$

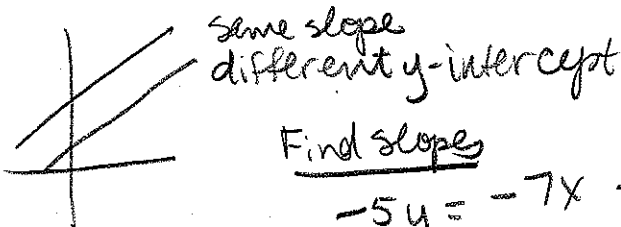
or

$$\boxed{(t, \frac{1}{2}t - 3)}$$

3. (a) Find the value of  $k$  so that the given system has no solution.

$$\begin{aligned} 7x - 5y &= -3 \\ 3x + ky &= 15 \end{aligned}$$

(no solution means parallel lines)



Find slopes

$$\begin{aligned} -5y &= -7x - 3 \\ y &= \frac{7}{5}x + \frac{3}{5} \\ m &= \frac{7}{5}, b = \frac{3}{5} \end{aligned}$$

$$\begin{aligned} ky &= -3x + 15 \\ y &= -\frac{3}{k}x + \frac{15}{k} \\ m &= -\frac{3}{k} \end{aligned}$$

$$\frac{7}{5} = -\frac{3}{k} \quad \rightarrow \quad k = -\frac{15}{7}$$

Check that y-intercepts are different:

$$\begin{aligned} b &= \frac{15}{k} = \frac{15}{-15/7} \\ &= \frac{15 \cdot 7}{-15} \\ &= -7 \neq \frac{3}{5} \end{aligned}$$

$k = -\frac{15}{7}$

(b) Is it possible to find a value of  $k$  so that the system has infinitely many solutions? Explain.

Infinitely many solutions means the equations in the system represent the same line, i.e. same slopes + same y-intercept. For same slope,  $k$  must be  $-\frac{15}{7}$ . ~~but~~ The y-intercept of the 1<sup>st</sup> line is  $\frac{3}{5}$ , but with  $k = -\frac{15}{7}$ , the y-intercept of the 2<sup>nd</sup> line is  $-7 \neq \frac{3}{5}$ . No, there is no  $k$  so that the lines are the same.

(c) For what value(s) of  $k$  will the system have a unique solution?

for any  $k \neq -\frac{15}{7}$

For the next 3 exercises, set up the system of equations but do not solve.

4. (#49, pg. 74 of *Finite Mathematics* by Lial, et. al.) The U-Drive Rent-A-Truck Co. plans to spend \$6 million on 200 new vehicles. Each van will cost \$20,000, each small truck \$30,000, and each large truck \$50,000. Past experience shows that they need twice as many vans as small trucks. How many of each kind of vehicle can they buy?

Let  $x$  = the number of vans to be purchased.  
 Let  $y$  = small trucks  
 Let  $z$  = large trucks

$$x + y + z = 200$$

$$20000x + 30000y + 50000z = 6000000$$

$$x = 2y$$

twice as many vans as small trucks  
 \*reword\*  
 the # of vans is 2 times the # of trucks  
 $x = 2y$

5. A cashier has a total of 96 bills in his register in one-, five-, and ten-dollar denominations. If he has three times as many fives as ones, and if the number of ones and fives combined is half of the number of tens he has, how many bills of each denomination does he have in his register?

Let  $x$  = the number of ones in the register.  
 Let  $y$  = fives  
 Let  $z$  = tens

$$x + y + z = 96$$

$$y = 3x$$

$$x + y = \frac{1}{2}z$$

The # of 5's is 3 \* the # of 1's  
 $y = 3x$

6. Random, Inc. makes picture collages in three sizes. A small collage requires 30 minutes of cutting time and 36 minutes of pasting time. A medium collage requires 60 minutes of cutting time and 54 minutes for pasting. A large collage requires 90 minutes for cutting and 72 minutes for pasting. There are 380 labor hours available for cutting and 330 labor hours available for pasting each week. If the company wants to run at full capacity, how many collages of each size should be made each week?

Let  $x$  = the number of small collages to be made each week  
 Let  $y$  = medium  
 Let  $z$  = large

*wants to make twice as many small collages as medium collages*

	small	medium	large	total avail
cutting time	30min	60min	90min	380 hrs * 60 = 22800 min
pasting time	36min	54min	72min	330 hrs * 60 = 19800 min

$$30x + 60y + 90z = 22800$$

$$36x + 54y + 72z = 19800$$

$$x = 2y$$

7. Let  $A = \begin{bmatrix} 4 & x \\ 3 & -7 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & -3 & 5 \\ 8 & 7 & -6 \end{bmatrix}$ ,  $C = \begin{bmatrix} 4 & -5 \\ 3 & 0 \\ 7 & -10 \end{bmatrix}$ , and  $D = \begin{bmatrix} 5 & 3 & -1 \\ 2 & 6 & 4 \end{bmatrix}$ . Compute each of the following:

- (a)  $B + 3D$
- (b)  $2C + B$
- (c)  $4D - 3C^T$
- (d)  $4a_{21} - 2c_{32} + 7d_{13}$
- (e)  $DB$
- (f)  $BC$
- (g)  $CD^T$
- (h)  $BB^T$
- (i)  $A^2$

a)  $B + 3D = \begin{bmatrix} 2 & -3 & 5 \\ 8 & 7 & -6 \end{bmatrix} + 3 \begin{bmatrix} 5 & 3 & -1 \\ 2 & 6 & 4 \end{bmatrix}$   
 $= \begin{bmatrix} 2 & -3 & 5 \\ 8 & 7 & -6 \end{bmatrix} + \begin{bmatrix} 15 & 9 & -3 \\ 6 & 18 & 12 \end{bmatrix}$   
 $= \begin{bmatrix} 17 & 4 & 2 \\ 14 & 25 & 6 \end{bmatrix}$

b)  $2C + B$   $C + B$  are not the same size, so we cannot add them.

c)  $4D - 3C^T = 4 \begin{bmatrix} 5 & 3 & -1 \\ 2 & 6 & 4 \end{bmatrix} - 3 \begin{bmatrix} 4 & 3 & 7 \\ -5 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 20 & 12 & -4 \\ 8 & 24 & 16 \end{bmatrix} - \begin{bmatrix} 12 & 9 & 21 \\ -15 & 0 & -30 \end{bmatrix}$   
 $= \begin{bmatrix} 8 & 3 & -25 \\ 23 & 24 & 46 \end{bmatrix}$

d)  $4a_{21} - 2c_{32} + 7d_{13} = 4(3) - 2(-10) + 7(-1) = 12 + 20 - 7 = \boxed{25}$

e)  $DB$   $\begin{matrix} \text{size of } D \\ 2 \times 3 \end{matrix}$   $\begin{matrix} \text{size of } B \\ 2 \times 3 \end{matrix}$  Not possible

f)  $BC$   $\begin{matrix} \text{size of } B \\ 2 \times 3 \end{matrix}$   $\begin{matrix} \text{size of } C \\ 3 \times 2 \end{matrix}$   $2 \times 2 = \text{size of } BC$   
 $BC = \begin{bmatrix} 2 & -3 & 5 \\ 8 & 7 & -6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 3 & 0 \\ 7 & -10 \end{bmatrix} = \begin{matrix} R_1 \rightarrow [R_1 C_1 & R_1 C_2] \\ R_2 \rightarrow [R_2 C_1 & R_2 C_2] \end{matrix} = \begin{bmatrix} 34 & -60 \\ 11 & 20 \end{bmatrix}$

$R_1 C_1 = \text{row 1 of } B \times \text{col 1 of } C$   
 $= [2 \ -3 \ 5] \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$   
 $= 8 - 9 + 35$   
 $= 34$

$R_1 C_2 = \text{row 1 of } B \times \text{col 2 of } C$   
 $[2 \ -3 \ 5] \begin{bmatrix} -5 \\ 0 \\ -10 \end{bmatrix}$   
 $= -10 - 50$   
 $= -60$

$R_2 C_1 = \text{row 2 of } B \times \text{col 1 of } C$   
 $= [8 \ 7 \ -6] \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$   
 $= 32 + 21 - 42$   
 $= 11$

$R_2 C_2 = \text{row 2 of } B \times \text{col 2 of } C$   
 $= [8 \ 7 \ -6] \begin{bmatrix} -5 \\ 0 \\ -10 \end{bmatrix}$   
 $= -40 + 60 = 20$

g)  $CD^T$

$$\begin{array}{cc} \text{size of } C & \text{size of } D^T \\ 3 \times 2 & 3 \times 2 \\ \uparrow & \uparrow \\ & \neq \end{array}$$

not possible

h)  $BB^T = \begin{bmatrix} 38 & -35 \\ -35 & 149 \end{bmatrix}$  (in calc)

i)  $A^2 = A \cdot A$

$$\begin{array}{cc} \text{size of } A & \text{size of } A \\ 2 \times 2 & 2 \times 2 \\ \uparrow & \uparrow \\ & = \\ & \uparrow \uparrow \\ & 2 \times 2 = \text{size of } A^2 \end{array}$$

$$AA = \begin{bmatrix} 4 & x \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 4 & x \\ 3 & -7 \end{bmatrix} = \begin{matrix} R_1 \rightarrow R_1 \times C_1 & C_1 \\ R_2 \rightarrow R_2 \times C_1 & C_2 \\ R_1 \times C_2 & \\ R_2 \times C_2 & \end{matrix} = \begin{bmatrix} 16+3x & 4x-7x \\ 12-21 & 3x+49 \end{bmatrix}$$

$$= \begin{bmatrix} 16+3x & -3x \\ -9 & 3x+49 \end{bmatrix}$$

8. Solve for x and y:

$$3x - 3y = -7$$

$$15y - 1 = -2x$$

$$15y = -2x + 1$$

$$y = -\frac{2}{15}x + \frac{1}{15}$$

$$3x - 3\left(-\frac{2}{15}x + \frac{1}{15}\right) = -7$$

$$3x + \frac{2}{5}x - \frac{1}{5} = -7$$

$$\frac{17}{5}x = -\frac{34}{5}$$

$$x = \frac{-34}{5} \cdot \frac{5}{17} = -2$$

$$y = -\frac{2}{15}(-2) + \frac{1}{15}$$

$$y = \frac{4}{15} + \frac{1}{15} = \frac{5}{15} = \frac{1}{3}$$

$$\boxed{\begin{matrix} x = -2 \\ y = \frac{1}{3} \end{matrix}}$$

$$3 \begin{bmatrix} 2 & x \\ 5y & -1 \end{bmatrix} - \begin{bmatrix} -6 & 1 \\ 3y & -5 \end{bmatrix}^T = \begin{bmatrix} 12 & -7 \\ -2x & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3x \\ 15y & -3 \end{bmatrix} - \begin{bmatrix} -6 & 3y \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ -2x & 2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 3x-3y \\ 15y-1 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ -2x & 2 \end{bmatrix}$$

9. The times (in minutes) required for assembling, testing, and packaging large and small capacity food processors are shown in the following table:

	Assembling	Testing	Packaging
Large	45	15	10
Small	30	10	5

(a) Define a matrix  $T$  that summarizes the above data.

$$T = \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix}$$

(b) Let  $M = \begin{bmatrix} 100 & 200 \end{bmatrix}$  represent the number of large and small food processors ordered, respectively. Find  $MT$  and explain the meaning of its entries.

$MT$   
 size of  $M$  size of  $T$   
 $1 \times 2 \quad \checkmark \quad 2 \times 3$   
 $\uparrow \quad \quad \quad \downarrow$   
 $1 \times 3 = \text{size of } MT$

$$MT = \begin{bmatrix} 100 & 200 \end{bmatrix} \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix}$$

$$= R_1 \begin{bmatrix} 100 \cdot 45 + 200 \cdot 30 & 100 \cdot 15 + 200 \cdot 10 & 100 \cdot 10 + 200 \cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10500 & 3500 & 7000 \end{bmatrix}$$

(c) If assembling costs \$3 per minute, testing costs \$1 per minute, and packaging costs \$2 per minute, find a matrix  $C$  that, when multiplied with  $T$ , gives the total cost for making each size of food processor.

$$C = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$TC = \begin{bmatrix} 45 & 15 & 10 \\ 30 & 10 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$= R_1 \begin{bmatrix} 110 \end{bmatrix} \leftarrow 45 \cdot 3 + 15 \cdot 1 + 10 \cdot 2$$

$$= R_2 \begin{bmatrix} 110 \end{bmatrix} \leftarrow 30 \cdot 3 + 10 \cdot 1 + 5 \cdot 2$$

Total amount of time spent assembling, testing, and packaging respectively for the entire order.

$TC$   
 size of  $T$  size of  $C$   
 $2 \times 3 \quad \checkmark \quad 3 \times 1$   
 $\uparrow \quad \quad \quad \downarrow$   
 $2 \times 1 = \text{size of } TC$

10. If  $A = \begin{bmatrix} -3 & 7 \\ 8 & 10 \end{bmatrix}$ , find  $A^{-1}$ .

In calc,  $A^{-1} = \begin{bmatrix} -5/43 & 7/46 \\ 4/43 & 3/86 \end{bmatrix}$

11. If  $B = \begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}$ , find  $B^{-1}$ :

$B$  is a singular matrix ( $B$  does not have an inverse)

12. Solve the following system of equations using matrix inverses.

$$\begin{aligned} 3x + 2y &= z + 2 \\ -3y + 2 &= -2x \\ x &= y + z + 4 \end{aligned}$$

$$3x + 2y - z = 2$$

$$2x - 3y = -2$$

$$x - y - z = 4$$

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix} \quad \text{so}$$

$$\boxed{\begin{aligned} x &= -1 \\ y &= 0 \\ z &= -5 \end{aligned}}$$