Math 166 - Week in Review #4

Section 2.1 - Multiplication Principle and Permutations

- If you wish to accomplish a big goal that requires intermediate steps and would like to know how many different ways there are to accomplish this big goal, simply list each of the individual steps required to meet this goal. Next to each step, write the number of ways that step can be done. To get the total number of ways of accomplishing the big goal, multiply all the numbers listed next to the individual steps. This is the multiplication principle.
- Formal Definition of the Generalized Multiplication Principle - Suppose that a task T_1 can be performed in n_1 ways, a task T_2 can be performed in n_2 ways, ..., and, finally, a task T_k can be performed in n_k ways. Then the number of ways of performing tasks $T_1, T_2, \ldots, T_k$ in succession is given by the product $n_1 n_2 \cdots n_k$.
- Permutation - Given a set of distinct objects, a *permutation* of the set is an arrangement of these objects in a definite order.
- For permutations, ORDER MATTERS.
- Factorial Notation: For any positive integer n, $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. For example, $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$.
- The factorial symbol ! can be found on the calculator by pressing MATH, arrowing over to PRB, and selecting option 4.
- Permutations of n Distinct Objects - The number of permutations of n distinct objects taken r at a time is $P(n,r) = \dfrac{n!}{(n-r)!}$.
- In the notation $P(n,r)$, the P stands for permutation, the n is the number of distinct objects that you are starting with, and the r is the number of those objects that you are arranging.
- The *nPr* command can be found in the calculator by pressing MATH, arrowing over to PRB, and selecting option #2.
- Permutations of n Objects, **Not All Distinct** - Given a set of n objects in which the first type of object is repeated n_1 times, the second type of object is repeated n_2 times, ... ,and, finally, the last type of object is repeated n_r times so that $n_1 + n_2 + \cdots + n_r = n$, then the number of permutations of these n objects taken n at a time is given by $\dfrac{n!}{n_1!n_2!\cdots n_r!}$.

Section 2.2 - Combinations

- Combination - A combination is a subset of objects chosen from a given set where the order in which the objects were chosen does not matter.
- For combinations, order DOES NOT MATTER.
- Combinations of n Objects - The number of ways of choosing r objects from n distinct objects is given by $C(n,r) = \dfrac{n!}{r!(n-r)!}$.
- In the notation $C(n,r)$, the C stands for combination, the n is the number of objects that you are starting with, and the r is the number of those objects that you are choosing to be in a subset.
- The *nCr* command can be found in the calculator by pressing MATH, arrowing over to PRB, and selecting option #3.

1. Jeffrey is trying to find the perfect engagement ring for his girlfriend at a local jewelry store, and the jeweler has informed him that he has many decisions to make. He must first decide on the metal to be used–either yellow gold, white gold, or platinum. Then he must decide on the setting. The jewelry store has three types of settings: a solitaire setting, a setting with sidestones, and a multiple-stone setting. Next, he must choose the shape of the main diamond. The options are round, princess, emerald, asscher, marquise, oval, pear, and heart. After selecting the shape of diamond, he must choose from four different sizes for the diamond. How many possible engagement rings are there for Jeffrey to choose from?

$\underset{\text{metal}}{3} \cdot \underset{\text{setting}}{3} \cdot \underset{\text{shape}}{8} \cdot \underset{\text{size}}{4} = \boxed{288}$

2. Bill and Sue and seven of their friends go to the movies. They all sit next to each other in the same row. How many ways can this be done if

_ _ _ _ _ _ _ _ _

(a) Sue and Bill must sit next to each other?

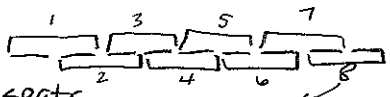
Method 1: Think of Sue & Bill as glued together

T_1 - Arrange the 7 with [Bill Sue] $n_1 = 8!$

T_2 - Now arrange Bill + Sue $n_2 = 2!$

$8! 2! = \boxed{80,640}$

method 2:



T_1 - Choose 2 seats side-by-side $n_1 = 8$ options

T_2 - Arrange Bill + Sue in their 2 chains $n_2 = 2!$

T_3 - Arrange the other 7 people $n_3 = 7!$

$8 \cdot 2! \cdot 7! = \boxed{80,640}$

(b) Sue must not sit next to Bill?

(Total # of arrangements) − (number of arrangements where Bill + Sue are next to each other)

$= 9! - 80,640 = \boxed{282,240}$

(c) Sue must sit in the middle seat?

$\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underset{\uparrow}{\underline{1}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = \boxed{40,320}$

Sue

(d) Sue sits on one end of the row and Bill sits on the other end of the row?

$\underset{\uparrow}{\underline{2}} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underset{\uparrow}{\underline{1}} = 2!7! = \boxed{10,080}$

Sue or Bill

(e) Sue, Bill, or Jan sits in the middle seat?

$\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underset{\uparrow}{\underline{3}} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 3 \cdot 8! = \boxed{120,960}$

Sue, Bill, or Jan

(f) Sue, Bill, and Jan sit in the middle three seats?

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underbrace{\underline{3} \cdot \underline{2} \cdot \underline{1}}_{\text{Sue, Bill, Jan}} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 3!6! = \boxed{4,320}$

3. Many U.S. license plates display a sequence of three letters followed by three digits.

(a) How many such license plates are possible?

$\underbrace{26\ 26\ 26}_{\text{letters}}\ \underbrace{10\ 10\ 10}_{\text{digits}} = \boxed{17576000}$

(b) How many of these have no repeated letters?

$\underbrace{26 \cdot 25 \cdot 24}_{\text{letters, no repeats}} \cdot \underbrace{10 \cdot 10 \cdot 10}_{\text{digits}} = \boxed{15{,}600{,}000}$

(c) How many license plates with three letters followed by three digits have exactly two letters that are the same?

Total − # of license plates with no repeats − # of plates with all 3 letters are the same.

$17{,}576{,}000 - 15{,}600{,}000 - 26 \cdot 1 \cdot 1 \cdot 10 \cdot 10 \cdot 10$

$= \boxed{1{,}950{,}000}$

(d) In order to avoid confusion of letters with digits, some states do not use the letters I, O or Q on their license plates. How many of these license plates are possible (again, three letters followed by three digits)?

$23 \cdot 23 \cdot 23 \cdot 10 \cdot 10 \cdot 10 = \boxed{12{,}167{,}000}$

(e) Assuming that the letter combinations VET, MDZ and DPZ are reserved for disabled veterans, medical practitioners, and disabled persons respectively, and also taking the restriction in part d into account, how many license plates are available for people who do not fall into one of those three categories?

Total − # of license plates with VET, MDZ, or DPZ.

$= 12{,}167{,}000 - \underbrace{3}_{\text{letters: VET or MDZ or DPZ; 3 options}} \cdot \underbrace{10 \cdot 10 \cdot 10}_{\text{digits}} = \boxed{12{,}164{,}000}$

4. Dripping wet after your shower, you have completely forgotten the combination of your lock. It is one of those "standard" combination locks, which uses a three number combination with each number in the range of 0 through 39. All you remember is that the second number is either 27 or 37, while the third number either is 5 or ends in a 5. In desperation, you decide to go through all possible combinations. Assuming that it takes about 10 seconds to try each combination, what is the longest possible time it can take to open your locker?

$40 \cdot 2 \cdot 4 = 320$ combinations

(2: 27 or 37; 4: 5, 15, 25, or 35)

$320 \text{ combos} * 10 \frac{\text{seconds}}{\text{combo}} = \boxed{3200 \text{ seconds}}$

5. How many 4-person committees are possible from a group of 9 people if:

(a) There are no restrictions?

$C(9,4) = \boxed{126}$

(b) Both Bob and Sam must be on the committee?

T_1 - Choose Bob $n_1 = 1$
T_2 - Choose Sam $n_2 = 1$
T_3 - Choose 2 others $n_3 = C(7,2)$
} multiply

sufficient

$C(7,2) = \boxed{21}$

(c) Bob or Sam is on the committee, but not both?

T_1 - Choose 1 of Sam and Bob $n_1 = C(2,1)$
T_2 - Choose 3 others (but not Sam or Bob) $n_2 = C(7,3)$

$C(2,1)C(7,3) = \boxed{70}$

6. A jewelry store chain with 8 stores in Georgia, 12 in Florida, and 10 in Alabama is planning to close 10 of these stores.

(a) How many ways can this be done?

$C(30,10) = 30{,}045{,}015$

(b) The company decided to close 2 stores in Georgia, 5 in Florida, and 3 in Alabama. How many ways can this be done?

T_1 - Choose 2 in Georgia $n_1 = C(8,2)$
T_2 - Choose 5 in Florida $n_2 = C(12,5)$
T_3 - Choose 3 in Alabama $n_3 = C(10,3)$

multiply!

$C(8,2)C(12,5)C(10,3) = \boxed{2{,}661{,}120}$

7. You have a box that contains 8 red, 7 black, 2 green, and 6 purple marbles. If you take a sample of six marbles from the box, how many ways can you get

(a) exactly 2 purple marbles and exactly 4 black marbles?

T_1 - choose 2 purple $n_1 = C(6,2)$

T_2 - choose 4 black $n_2 = C(7,4)$

$C(6,2)C(7,4) = \boxed{525}$

(b) exactly 2 purple marbles or exactly 4 black marbles? A - set of samples with exactly 2 purple

union, not mutually exclusive B - set of samples with exactly 4 black.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= C(6,2)C(17,4) + C(7,4)C(16,2) - C(6,2)C(7,4)$

(Exactly 2 Purple; Exactly 4 Black; Exactly 2 Purple AND 4 Black)

$= \boxed{39,375}$

(c) exactly 2 red marbles or exactly 1 green marble? E - set of samples with exactly 2 red

union, not mutually exclusive F - set of samples with exactly 1 green

$n(E \cup F) = n(E) + n(F) - n(E \cap F)$

$= C(8,2)C(15,4) + C(2,1)C(21,5) - C(8,2)C(2,1)C(13,3)$

(Exactly 2 Red; Exactly 1 green; Exactly 2 red and 1 green)

$= \boxed{62902}$

(d) at least 4 red marbles? (4 or more) Mutually Exclusive!

Exactly 4 Red or Exactly 5 Red or Exactly 6 Red

$C(8,4)C(15,2) + C(8,5)C(15,1) + C(8,6)$

$= \boxed{8218}$

(e) at most 4 purple marbles? one method:

(4 or less so Exactly 4 or 3 or 2 or 1 or 0. (similar to work above)

Another Method: Total - what we don't want.

All samples of 6 - samples w/Exactly 5 purple - samples with Exactly 6 purple

$C(23,6) - C(6,5)C(17,1) - C(6,6)$

$= \boxed{100844}$

8. Stan is having a mixed stroke of luck. He just got the phone number of his waitress, but he cannot read her handwriting. If he is certain that the first digit is a 5, the fourth digit is a 2 or a 7, and the last digit is a 6, what is the maximum number of phone numbers Stan must try? (Assume the phone number has only 7 digits.)

$1 \cdot 10 \cdot 10 \cdot 2 \cdot 10 \cdot 10 \cdot 1 = \boxed{20{,}000}$

(1: a 5; 2: 2 or 7; 1: a 6)

9. The state Motor Vehicle Department requires learners to pass a written test on the motor vehicle laws of the state. The exam consists of ten true/false questions, of which at least eight must be answered correctly to qualify for a permit. In how many different ways can a person who answers all the questions on the exam qualify for a permit?

Need at least 8 correct.

mutually exclusive so just add!

Exactly 8 or Exactly 9 or Exactly 10

$C(10,8)\cdot 1^8\cdot 1^2 + C(10,9)1^9\cdot 1^1 + C(10,10)\cdot 1^{10} = C(10,8)+C(10,9)+C(10,10)$

$= \boxed{56}$

10. In a different state, the Motor Vehicle Department requires learners to pass a similar test with 10 multiple choice questions, of which at least 8 must be answered correctly to qualify for a permit. If each question has 4 choices, in how many different ways can a person who answers all the questions on the exam qualify for a permit?

For Exactly 8 Correct

T_1 - Choose 8 to be correct $n_1 = C(10,8)$ (circled blanks)

T_2 - Answer these correctly $n_2 = 1^8$ (product of 1's in circled blanks)

T_3 - Choose others to be wrong $n_3 = C(2,2)$

T_4 - Answer these incorrectly $n_4 = 3^2$

(1) 3 (1) (1) (1) 3 (1) (1) (1) (1) (wrong)

at least 8 correct = 8 or more

mutually exclusive!

Exactly 8 or Exactly 9 or Exactly 10

$C(10,8)\cdot 1^8\cdot 3^2 + C(10,9)\cdot 1^9\cdot 3^1 + C(10,10)\cdot 1^{10}$

$= \boxed{436}$

11. How many different arrangements can be made from the letters of MASSACHUSETTS?

$\frac{13!}{1!2!4!1!1!1!1!2!} = \frac{13!}{2!4!2!} = \boxed{64{,}864{,}800}$

12. Jane has 3 yellow pillows, 6 purple pillows, 8 red pillows and 2 green pillows. In how many ways can Jane line up these pillows in a single row on her couch if pillows of the same color are identical?

permutation

↑ nondistinct objects

$\frac{19!}{3!6!8!2!} = \boxed{349{,}188{,}840}$

13. Six boyfriend-girlfriend couples attend a party. They hire a photographer to take their picture. In how many ways can the group line up for the picture (in one row) if

permutation

(a) there are no restrictions?

$P(12,12) = \boxed{12!}$

(b) couples must stand together?

T_1 - arrange the 6 couples $n_1 = 6!$

T_2 - arrange the 2 people in each couple $n_2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$

$6! \cdot 2^6 = \boxed{46,080}$

(c) they decide to take the picture with only 4 people in it?

$P(12,4) = \underline{12} \cdot \underline{11} \cdot \underline{10} \cdot \underline{9} = \boxed{11,880}$

(d) they decide to take the picture with only 2 *couples* in them? (Assume that there are no restrictions on the order the people stand in line for this picture.)

T_1 - Choose 2 couples $n_1 = C(6,2)$

T_4 - arrange these 4 people $n_2 = 4!$

$C(6,2)4! = \boxed{360}$

14. Anabel, Brayden, Callie, Domingo, Emma, and Frank go to the movies and all sit in the first row, which has 10 seats available. How many different seating arrangements of these six friends are possible in these 10 seats?

_ _ _ _ _ _ _ _ _ _

T_1 - Choose 6 seats $n_1 = C(10,6)$

T_2 - arrange the 6 in the chosen seats $n_2 = 6!$

$C(10,6) \cdot 6! = \boxed{151,200}$

15. Five cards are randomly selected from a standard deck of 52 playing cards. How many 5-card hands are possible

(a) if there are no restrictions?

$C(52,5) = 2,598,960$

(b) that include two pairs?

T_1 - choose 2 ranks for the 2 pairs $n_1 = C(13,2)$

T_2 - choose 2 cards of one rank $n_2 = C(4,2)$

T_3 - choose 2 cards of other rank $n_3 = C(4,2)$

T_4 - Choose one other card, dif. rank $n_4 = C(44,1)$

$C(13,2)C(4,2)C(4,2)C(44,1) = \boxed{123,552}$

(c) that have exactly one pair?

T_1 - choose rank of pair $n_1 = C(13,1)$

T_2 - Choose 2 of that rank $n_2 = C(4,2)$

T_3 - Choose 3 other ranks $n_3 = C(12,3)$

T_4 - Choose 1 card of one of the other ranks $n_4 = C(4,1)$

T_5 - Choose 1 of another chosen rank $n_5 = C(4,1)$

T_6 - Choose 1 of last chosen rank $n_6 = C(4,1)$

$C(13,1)C(4,2)C(12,3)(C(4,1))^3 = \boxed{1,098,240}$

(d) for a full house? (A full house is a hand with three cards of one rank and two cards of another rank.)

T_1 - Choose rank for 3 of a kind $n_1 = C(13,1)$

T_2 - Choose 3 of this rank $n_2 = C(4,3)$

T_3 - Choose rank for pair $n_3 = C(12,1)$

T_4 - Choose 2 of this rank $n_4 = C(4,2)$

$C(13,1)C(4,3)C(12,1)C(4,2) = \boxed{3,744}$