Math 141 - Week in Review #4

1. Graph the solution set for the given system of linear inequalities:

$$\begin{aligned} x+y &\leq 36 \\ 10 \leq x &\leq 30 \\ 6 \leq y &\leq 20 \end{aligned}$$

L_1 $x+y=36$ (solid line)

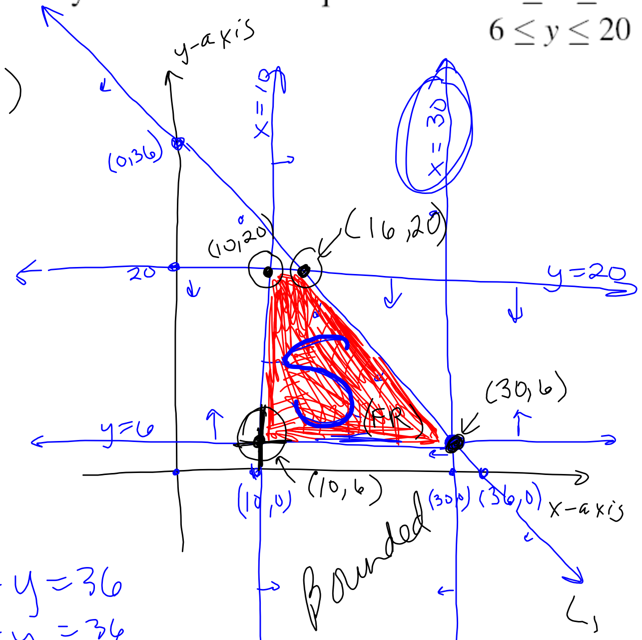
when $x=0$, $y=36$ $(0,36)$

when $y=0$, $x=36$ $(36,0)$

Test $(0,0)$

$x+y\leq 36$

$0+0\leq 36$? ✓



$10\leq x\leq 30$

$x\geq 10$ AND $x\leq 30$

$x=10$ $x=30$

vertical, solid

$x+y=36$

$30+y=36$

$y=6$

L_1 and $x=30$

$x+y=36$

$x+20=36$

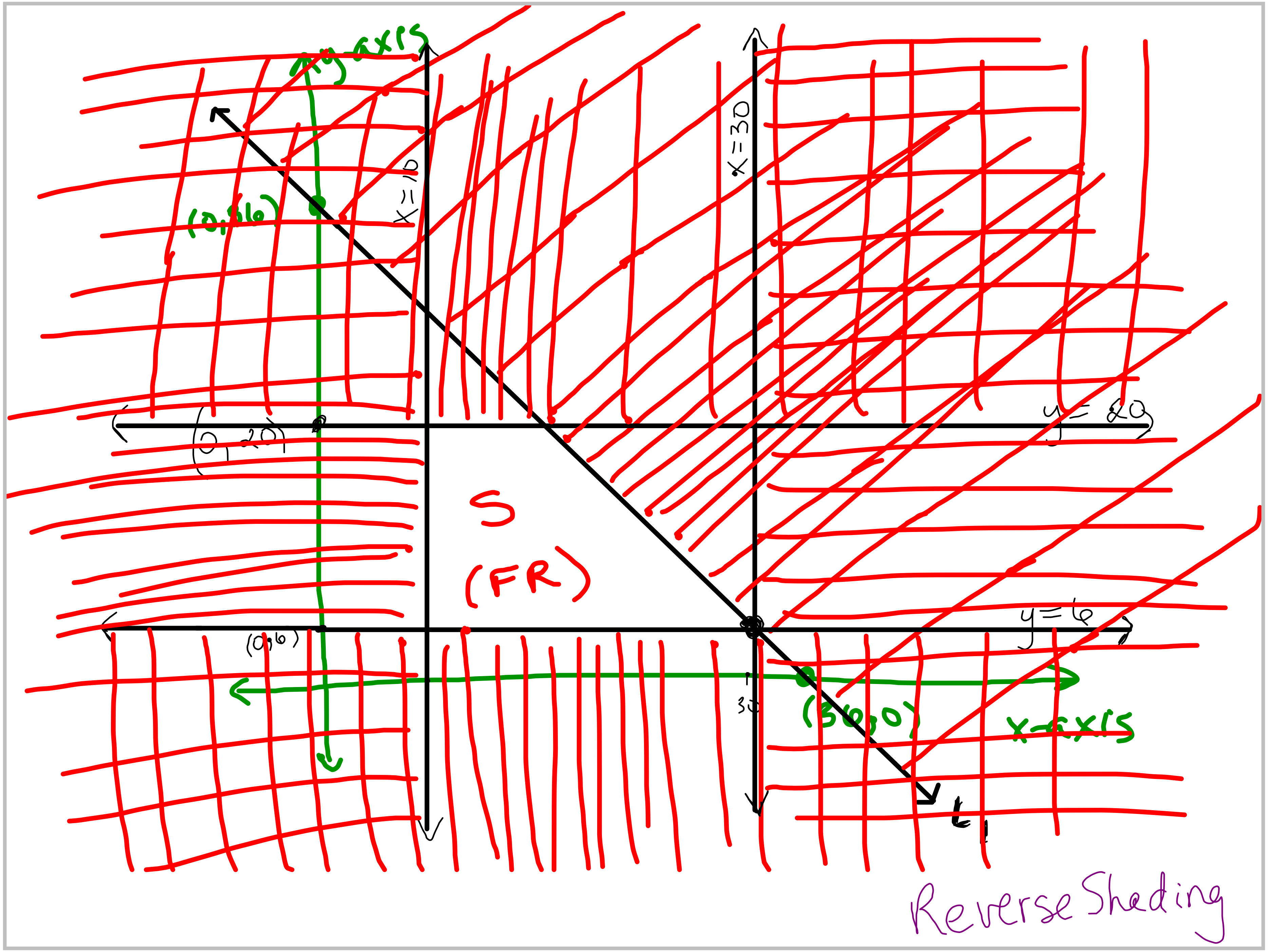
$x=16$

$6\leq y\leq 20$

$y\geq 6$ AND $y\leq 20$

$y=6$ $y=20$

horizontal, solid.


y-axis
x = 10
x = 30
(0, 36)
(0, 20)
y = 20
S
(FR)
(0, 6)
y = 6
36
(36, 0)
x-axis
L1
Reverse Shading

2. Find the maximum and minimum values (and their locations) for the function $G = 3x + 5y$ subject to the constraints given in #1.

maximize + minimize $G = 3x + 5y$

Method of Corners

Step 1 - Graph the feasible region (#1)

Steps 2+3:

	Corners	Value of $G = 3x + 5y$
	(30, 6)	$3(30) + 5(6) = 120$
min	(10, 6)	$3(10) + 5(6) = 60$
	(10, 20)	$3(10) + 5(20) = 130$
max	(16, 20)	$3(16) + 5(20) = 148$

The maximum value of G is 148 and occurs when $x = 16$ and $y = 20$.

The minimum value of G is 60 and occurs when $x = 10$ and $y = 6$.

For the next 3 exercises, first solve by the Method of Corners, and then solve by the Simplex Method (from Section 4.1) if possible.

3. A bicycle manufacturer makes a 3-speed and a 10-speed model with two operations: assembly and painting. Each 3-speed bike requires 1 hour to assemble and 2 hours to paint. Each 10-speed bike takes 1 hour to assemble and 1 hour to paint. The assembly operation has 80 hours per week available, and the painting operation has 100 work hours available each week. If the company makes $80 on each 3-speed bike and $60 on each 10-speed bike, how many bicycles of each type should be made and sold each week to maximize profits? (#39, pg. 226 of *Finite Mathematics: An Applied Approach* by Young, et. al.)

(1) Let x = the number of 3-speed bikes to be made + sold.
Let y = - - - - 10-speed - - - - -

	3 speed bikes	10 speed bikes	Amt Avail
assembly time	1 hr	1 hr	80 hr
painting time	2 hr	1 hr	100 hr
profit per unit	$80	$60	

(2) State the objective + the objective function.

Maximize Profit $P = 80x + 60y$

(3) Constraints

subject to:

0 or bigger

$$\boxed{\begin{aligned} x + y &\le 80 \\ 2x + y &\le 100 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}}$$

(4) Solve.

Method of Corners

(1) Graph the constraints.

L_1 $x+y=80$ (solid)

$(0,80)$

$(80,0)$

$x+y \le 80$

$y \le -x+80$

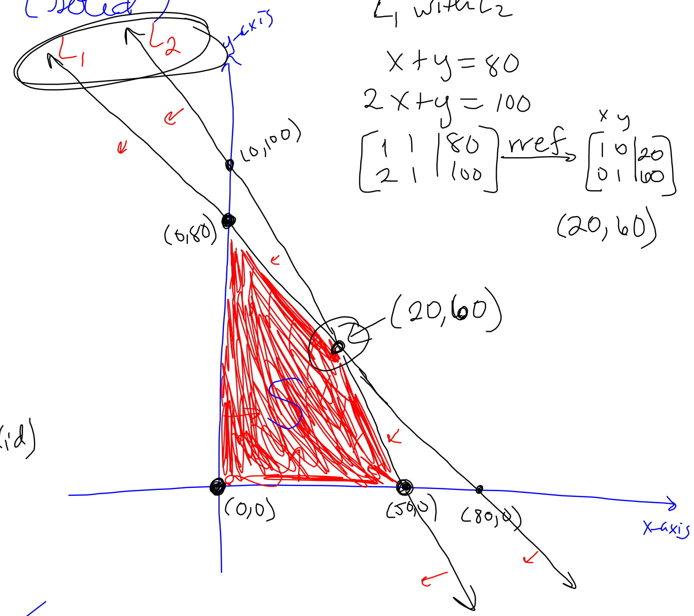
L_2 $2x+y=100$ (solid)

$(0,100)$, $(50,0)$

Test $(0,0)$

$2(0)+0 \le 100$? ✓

$x \ge 0$, $y \ge 0$



L_1 with L_2

$x+y=80$

$2x+y=100$

$$\left[\begin{array}{cc|c} 1 & 1 & 80 \\ 2 & 1 & 100 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 20 \\ 0 & 1 & 60 \end{array}\right] \end{array}$$

$(20,60)$

Corners	Value of $P = 80x+60y$
$(0,80)$	$80(0)+60(80)=4800$
$(50,0)$	4000
$(0,0)$	0
$(20,60)$	5200

The maximum value of P is \$5200 and occurs when $x=20$ and $y=60$.

Profit is \$5200

20 3-speed bikes

60 10-speed bikes.

Maximize $P = 80x + 60y$

Subject to
$$x + y \le 80$$
$$2x + y \le 100$$
$$x \ge 0,\ y \ge 0.$$

$$x + y + u \qquad = 80$$
$$2x + y \qquad + v \qquad = 100$$
$$-80x - 60y \qquad + P = 0$$

1st pivot element

$$\begin{array}{cccccc} x & y & u & v & P & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 80 \\ 2 & 1 & 0 & 1 & 0 & 100 \\ \hline -80 & -60 & 0 & 0 & 1 & 0 \end{array}\right]$$

Initial simplex tableau.

$80/1 = 80$

$100/2 = 50 \leftarrow$

$x = 0 \quad u = 80 \quad P = 0$

$y = 0 \quad v = 100$

1st Pivot R 2 C 1

Matrix after 1st pivot

2nd pivot

$$\begin{array}{cccccc} x & y & u & v & P & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 1/2 & 1 & -1/2 & 0 & 30 \\ 1 & 1/2 & 0 & 1/2 & 0 & 50 \\ \hline 0 & -20 & 0 & 40 & 1 & 4000 \end{array}\right]$$

$30/\frac{1}{2} = 60 \leftarrow$

$50/\frac{1}{2} = 100$

R 1 C 2

After the 2nd pivot

$$\begin{array}{cccccc} x & y & u & v & P & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & -1 & 0 & 60 \\ 1 & 0 & -1 & 1 & 0 & 20 \\ \hline 0 & 0 & 40 & 20 & 1 & 5200 \end{array}\right]$$

Final simplex tableau.

$x = 20 \quad u = 0 \quad P = 5200$

$y = 60 \quad v = 0$

$\leftarrow$ slack variables represent left over resources.

The maximum value of $P = \$5200$ and occurs when $x = 20$ and $y = 60$.

(no left over resources)

4. A company makes two calculators: a business model and a scientific model. The business model contains 10 microcircuits and requires 20 minutes to program, while the scientific model contains 20 microcircuits and requires 30 minutes to program. The company has a contract that requires it to use at least 320 microcircuits each day, and the company has 14 hours of programming time available each day. The company also wants to make at least twice as many business calculators as scientific calculators. If each business calculator requires 10 production steps and each scientific calculator reqires 12 production steps, how many calculators of each type should be made each day to minimize the number of production steps? (adapted from #24, pg. 192 of *Finite Mathematics: An Applied Approach* by Young, et. al.)

Let x = the number of business calculators made each day.

Let y = the number of scientific calculators made each day.

	Business Calcs	Scientific Calcs	
# of micro~~processors~~circuits	10	20	320 ← min required
programming time	20 min	30 min	14 hrs available 14*60 = 840 min
# of production steps	10	12	

Minimize the Number of Production Steps

Minimize $N = 10x + 12y$

objective function

Subject to

$$10x + 20y \geq 320$$
$$20x + 30y \leq 840$$
$$x \geq 2y$$
$$x \geq 0$$
$$y \geq 0$$

$10x + 20y \geq 320$

$20x + 30y \leq 840$

$x \geq 2y$

$x \geq 0$

$y \geq 0$

L_1: $10x + 20y = 320$

$(0,16)$ $(32,0)$

Test $(0,0)$

$10(0) + 20(0) \geq 320$?

FALSE

$20x + 30y = 840$

$x - 2y = 0$

$\left[\begin{array}{cc|c} 20 & 30 & 840 \\ 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 24 \\ 0 & 1 & 12 \end{array}\right]$

L_2

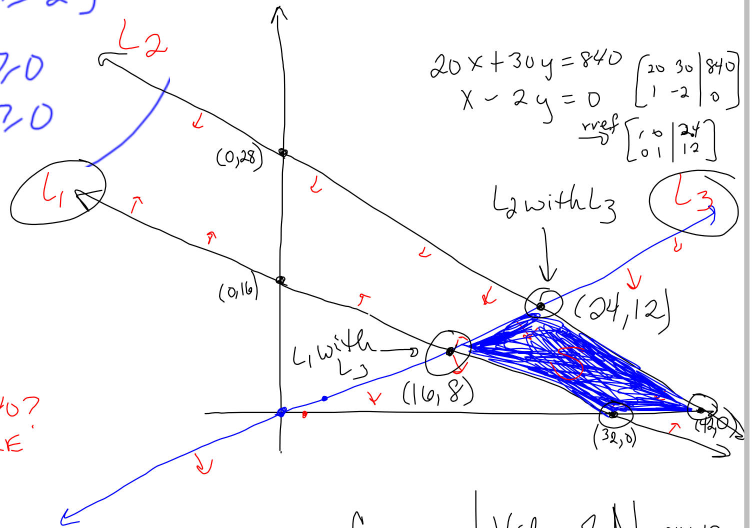
$20x + 30y = 840$

$(0,28)$ $(42,0)$

Test $(0,0)$

$20(0) + 30(0) \leq 840$?

TRUE



L_3: $x = 2y$

$(0,0)$ When $y = 2$, then

$x = 2(2) = 4$ $(4,2)$

Test $(1,0)$

$x \geq 2y$

$1 \geq 2(0)$

$1 \geq 0$? TRUE

Corners	Value of $N = 10x + 12y$
$(32,0)$	320
$(42,0)$	420
$(24,12)$	384
$(16,8)$	256

The minimum number of steps is 256 and occurs when we make 16 business calculators and 8 scientific calculators.

5. The directors of a state fair want to bring in entertainers to bolster attendance at the fair. They find that a top star demands \$10,000, a faded star \$6,000, and high-quality local talent \$3,000 for each performance. The directors estimate that 8,000 people will attend a top-star performance, 3,000 will attend a faded-star performance, and 1,200 will attend a high-quality local talent performance. The total amount to be spent for such entertainers is not to exceed \$50,000, and the directors decide to contract for no more than six performers in total. The amount spent on advertising is not to exceed \$4,000, and the directors estimate that the advertising costs for each type of entertainer will be \$500, \$400, and \$250, respectively. Assuming one performance for each entertainer, how many of each type should be contracted to maximize attendance? Will there be any surplus contracting or advertising funds left over? If so, how much?

(#49, pg. 227 of *Finite Mathematics: An Applied Approach* by Young, et. al.)

Let x equal the number of top stars to be contracted
Let y - - - - - - faded stars - - - .
Let z - - - - - - local talent performers - - - .

	top star	faded star	local talent	
Cost per performance	\$10000	\$6000	\$3000	\$50000 ← limit
attendance	8000	3000	1200	
Advertising cost	\$500	\$400	\$250	\$4000 ← limit

Maximize Attendance $A = 8000x + 3000y + 1200z$

subject to
$$10000x + 6000y + 3000z \le 50000$$
$$500x + 400y + 250z \le 4000$$
$$x + y + z \le 6$$
$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

Maximize Attendance $A = 8000x + 3000y + 1200z$

subject to
$$10000x + 6000y + 3000z \le 50000$$
$$500x + 400y + 250z \le 4000$$
$$x + y + z \le 6$$
$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

$10000x + 6000y + 3000z + u = 50000$ ← leftover performance \$

$500x + 400y + 250z + v = 4000$ ← left over advertising \$

$x + y + z + w = 6$

$-8000x - 3000y - 1200z + A = 0$

1st pivot element

x	y	z	u	v	w	A		
10000	6000	3000	1	0	0	0	50000	50000/10000 = 5 ←
500	400	250	0	1	0	0	4000	4000/500 = 8
1	1	1	0	0	1	0	6	6/1 = 6
-8000	-3000	-1200	0	0	0	1	0	

4x8

R1 C1

After 1st Pivot

x	y	z	u	v	w	A	
1	3/5	3/10	.0001	0	0	0	5
0	100	100	-1/20	1	0	0	1500
0	2/5	7/10	-.0001	0	1	0	1
0	1800	1200	4/5	0	0	1	40000

Done!

$x = 5$ $\quad u = 0$ ← slack variables

$y = 0$ $\quad v = 1500$ $\quad A = 40000$

$z = 0$ $\quad w = 1$

The maximum attendance is 40000 and occurs when 5 top star performers, 0 faded stars, and 0 local talent performers are hired.

Since $v = 1500$, there is \$1500 left in the advertising account.

For the next two exercises, first solve using the Method of Corners, and then solve using the Simplex Method.

6. Maximize $P = 2x + 5y$
subject to
$$\begin{aligned} -3x + y &\le 6 \\ x - y &\le 7 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Method of Corners

1) Graph the feasible set.

L_1: $-3x+y=6$

(0,6) (−2,0)

Test (0,0)

$0+0 \le 6$? ✓

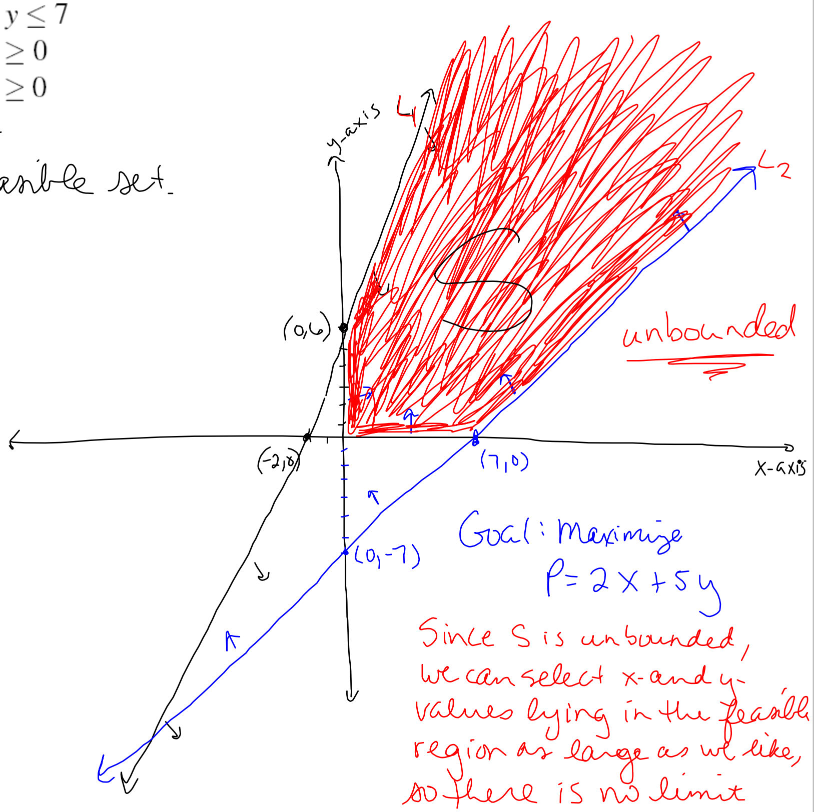
L_2: $x-y=7$

(0,−7) (7,0)

Test (0,0)

$0-0 \le 7$ ✓

$x \ge 0, y \ge 0$ ← QI



Goal: Maximize
$P = 2x + 5y$

Since S is unbounded, we can select x- and y-values lying in the feasible region as large as we like, so there is no limit to how big we can make P. This means that P has no maximum value.

6. Maximize $P = 2x + 5y$
subject to
$$-3x + y \le 6$$
$$x - y \le 7$$
$$x \ge 0$$
$$y \ge 0$$

Simplex Method

$$-3x + y + u = 6$$
$$x - y + v = 7$$
$$-2x - 5y + P = 0$$

$$\begin{array}{c} \begin{array}{ccccc} x & y & u & v & P \end{array} \\ \left[\begin{array}{ccccc|c} -3 & 1 & 1 & 0 & 0 & 6 \\ 1 & \cancel{-1} & 0 & 1 & 0 & 7 \\ \hline -2 & -5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

(can't pivot on negative numbers)

1st pivot: R1 C2

After 1st pivot:

$$\left[\begin{array}{ccccc|c} \cancel{-3} & 1 & 1 & 0 & 0 & 6 \\ \cancel{-2} & 0 & 1 & 1 & 0 & 13 \\ \hline -17 & 0 & 5 & 0 & 1 & 30 \end{array}\right]$$

can't pivot on negative numbers.

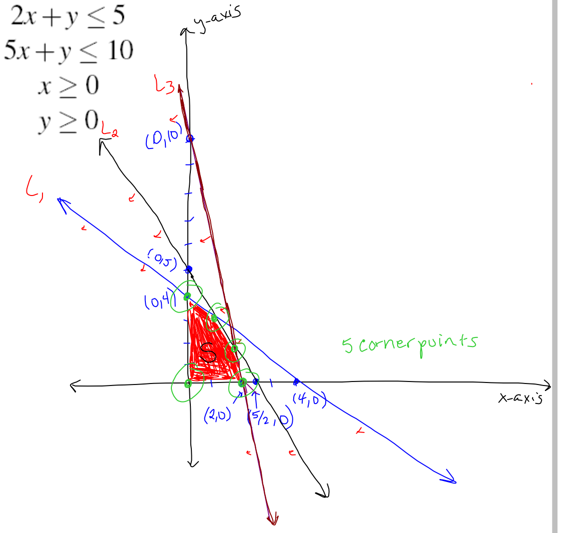
Negative in last row means maximum value has not been reached, but since we cannot pivot on a negative number, no other pivots are possible. This means there is no solution, so P does not have a maximum value.

7. Maximize $P = 2x + y$
subject to
$$x + y \le 4$$
$$2x + y \le 5$$
$$5x + y \le 10$$
$$x \ge 0$$
$$y \ge 0$$

L_1
$x+y=4$
$(0,4)$ $(4,0)$
Test $(0,0)$
$0+0 \le 4$ ✓

L_2
$2x+y=5$
$(0,5)$ $(\frac{5}{2},0)$
Test $(0,0)$
$2(0)+0 \le 5$ ✓

L_3
$5x+y=10$
$(0,10)$ $(2,0)$
Test $(0,0)$
$5(0)+0 \le 10$ ✓



Corners	Value of $P = 2x+y$
$(0,0)$	0
$(0,4)$	4
$(2,0)$	4
$(1,3)$	5
$(\frac{5}{3},\frac{5}{3})$	5

max value

The maximum value of P is 5 and occurs at every point on the line segment joining $(1,3)$ to $(\frac{5}{3},\frac{5}{3})$.

7. Maximize $P = 2x + y$
subject to
$$x + y \le 4$$
$$2x + y \le 5$$
$$5x + y \le 10$$
$$x \ge 0$$
$$y \ge 0$$

Simplex Method

$$\begin{aligned} x + y + u \quad\quad &= 4 \\ 2x + y \quad + v \quad &= 5 \\ 5x + y \quad\quad + w &= 10 \\ -2x - y \quad\quad + P &= 0 \end{aligned}$$

Initial Simplex Tableau:

$$\begin{array}{cccccc|c} x & y & u & v & w & P & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 4 \\ 2 & 1 & 0 & 1 & 0 & 0 & 5 \\ 5 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array} \quad \begin{array}{l} \\ 4/1 = 4 \\ 5/2 = 2.5 \\ 10/5 = 2 \leftarrow \text{smallest ratio} \\ \\ \end{array}$$

1st pivot element

most negative

1st Pivot: R3 C1

After 1st Pivot

$$\begin{array}{cccccc|c} x & y & u & v & w & P & \\ \hline 0 & 4/5 & 1 & 0 & -1/5 & 0 & 2 \\ 0 & 3/5 & 0 & 1 & -2/5 & 0 & 1 \\ 1 & 1/5 & 0 & 0 & 1/5 & 0 & 2 \\ \hline 0 & -3/5 & 0 & 0 & 2/5 & 1 & 4 \end{array} \quad \begin{array}{l} \\ 2/\frac{4}{5} = 2.5 \\ 1/3/5 = 1.667 \leftarrow \\ 2/1/5 = 10 \\ \\ \end{array}$$

2nd Pivot: R2 C2

After 2nd Pivot

$$\begin{array}{cccccc|c} x & y & u & v & w & P & \\ \hline 0 & 0 & 1 & -4/3 & 1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 5/3 & -2/3 & 0 & 5/3 \\ 1 & 0 & 0 & -1/3 & 1/3 & 0 & 5/3 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 5 \end{array}$$

Since there are no negative numbers in the last row, we know that we have found the maximum value of P. (cont'd)

Reading the values of all the variables from this last matrix, we have

$x = 5/3$ $\quad u = 2/3$ $\quad P = 5$

$y = 5/3$ $\quad v = 0$

$w = 0$

so it looks like we are ready to conclude that the maximum value of P is 5 and occurs when $x = \frac{5}{3}$ and $y = \frac{5}{3}$. However, when we solved this problem using the Method of Corners, we saw that the maximum occurs at two corner points of the feasible set, meaning that the maximum value of P is 5 and occurs at **every point on the line segment** connecting $(\frac{5}{3}, \frac{5}{3})$ to $(1, 3)$. So the question is, **how do we know that there are infinitely many solutions on a line segment when we use the Simplex Method?**

ANSWER →

After the 2^{nd} pivot, we had obtained the following matrix:

$$\begin{array}{cccccc|c} x & y & u & v & w & P & \\ \hline 0 & 0 & 1 & -4/3 & 1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 5/3 & -2/3 & 0 & 5/3 \\ 1 & 0 & 0 & -1/3 & 1/3 & 0 & 5/3 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 5 \end{array}$$

The bottom row does not contain any negative numbers, but consider the 0's that are in the bottom row. The 1^{st} three 0's are in unit columns associated with x, y, and z. The 4^{th} 0 is in the w-column, a non-unit column. (**Note: Variables associated with non unit columns are called nonbasic variables and are given the value 0.**) **When you have a 0 at the bottom of a non-unit column, you may use that column as a pivot column.** Here's why: the bottom row of the simplex tableau actually represents the equation

$$0x + 0y + 0u + v + 0w + P = 5,$$

or equivalently,

$$P = -v + 5$$

If I pivot in the w column, then I will be changing the value of w. Since w does not appear in the current function for profit ($P = -v + 5$), changing w will not decrease profit.

So, take column 5 as the next pivot column and find the pivot row by computing the appropriate ratios.

Since $\frac{2}{3}/\frac{1}{3} = 2$ and $\frac{5}{3}/\frac{1}{3} = 5$ **(REMEMBER: we <u>Never</u> pivot on a 0 or on a negative number)**, we will use Row 1 for the pivot row.

3^{rd} Pivot: R1C5

After 3^{rd} Pivot

$$
\begin{array}{c}
\begin{array}{cccccc} x & y & u & v & w & P \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 0 & 3 & -4 & 1 & 0 & 2 \\
0 & 1 & 2 & -1 & 0 & 0 & 3 \\
1 & 0 & -1 & 1 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 1 & 5
\end{array}\right]
\end{array}
$$

Again there are no negatives in the bottom row, so we know the maximum value of P has been found. However, this time we have

$x = 1$ $\quad u = 0$ $\quad P = 5$

$y = 3$ $\quad v = 0$

$w = 2$

So, after the 3^{rd} pivot, we have arrived at a second corner point of the feasible region where we have the same maximum value of P (namely P=5). Just as in the method of corners, we can conclude that the maximum value of P is 5 and occurs **at every point on the line segment connecting $(\frac{5}{3}, \frac{5}{3})$ to $(1,3)$.**

In fact, if you perform a fourth pivot you will be sent back to the point $(\frac{5}{3}, \frac{5}{3})$ and still have P=5. To select the 4^{th} pivot column, look for 0's at the bottom of nonunit columns (since we know there are no negative numbers to look for in the last row). In the matrix above, we see that u has a non-unit column with a 0 at the bottom. This means we pivot in C3.

Select the pivot row by performing the ratio check:

$2/3 \approx 0.67$ and $3/2 = 1.5$, so the smallest ratio is in Row 1. (Remember, the -1 in Row 3 is not even considered because we cannot pivot on negative numbers.)

<u>4th Pivot</u>: R1C3

<u>After 4th Pivot</u>

$$\begin{array}{cccccc|c} x & y & u & v & w & P & \\ \hline 0 & 0 & 1 & -4/3 & 1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 5/3 & -2/3 & 0 & 5/3 \\ 1 & 0 & 0 & -1/3 & 1/3 & 0 & 5/3 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 5 \end{array}$$

This is the same exact matrix as we got after the 2nd pivot, so we see that we have been sent back to where $x = \frac{5}{3}$ and $y = \frac{5}{3}$ and P is still 5.