Math 142 - Week in Review #5

1. Suppose an object moves along the y-axis so that its location is $y = 3x^2 + 5x$ at time x, where y is measured in meters and x is measured in seconds.

   (a) Find the average rate of change of y with respect to x from $\overset{x=a}{x = 4}$ to $\overset{x=b}{x = 9}$. Interpret your answer.

$$\text{Avg. rate of change} = \frac{f(b)-f(a)}{b-a} = \frac{f(9)-f(4)}{9-4} = \frac{288-68}{5} = 44 \text{ m/s}$$

Between $x=4$ seconds and $x=9$ seconds, the object is traveling at an average speed of 44 meters per second.

   (b) Using the limit definition, find the instantaneous rate of change of y at $x = 4$. Interpret your answer.

Goal: $\lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h}$

① $f(4+h) = 3(4+h)^2 + 5(4+h)$
$= 3(16+8h+h^2)+20+5h$
$= 48+24h+3h^2+20+5h$
$= 3h^2+29h+68$

② $f(4+h)-f(4) = (3h^2+29h+68)-(68)$
$= 3h^2+29h$

③ $\dfrac{f(4+h)-f(4)}{h} = \dfrac{3h^2+29h}{h}$
$= 3h+29$

④ $\lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h}$
$= \lim\limits_{h\to 0} (3h+29)$
$= 29$ m/s

At exactly $x = 4$ seconds, the object is traveling at a speed of 29 m/s.

   (c) Confirm your answer to (b) using basic differentiation properties.

$f'(x) = 6x+5$

$f'(4) = 6(4)+5 = \boxed{29 \text{ m/s}}$

2. Let $f(x)=\sqrt{x-5}+3$.

(a) Use the limit definition of the derivative to find $f'(x)$.

① $f(x+h)=\sqrt{x+h-5}+3$

② $f(x+h)-f(x)=\left(\sqrt{x+h-5}+3\right)-\left(\sqrt{x-5}+3\right)$

$=\sqrt{x+h-5}-\sqrt{x-5}$

③ $\dfrac{f(x+h)-f(x)}{h}=\left(\dfrac{\sqrt{x+h-5}-\sqrt{x-5}}{h}\right)\left(\dfrac{\sqrt{x+h-5}+\sqrt{x-5}}{\sqrt{x+h-5}+\sqrt{x-5}}\right)$

$=\dfrac{(x+h-5)-(x-5)}{h\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$

$=\dfrac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-5}+\sqrt{x-5}\right)}$

$=\dfrac{1}{\sqrt{x+h-5}+\sqrt{x-5}}$

④ $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{x+h-5}+\sqrt{x-5}}$

$=\dfrac{1}{\sqrt{x-5}+\sqrt{x-5}}$

$=\dfrac{1}{2\sqrt{x-5}}$ so $\boxed{f'(x)=\dfrac{1}{2\sqrt{x-5}}}$

(b) Find the slope of the tangent line to $f(x)$ at $x=21$.

$m=f'(21)=\dfrac{1}{2\sqrt{21-5}}=\boxed{\dfrac{1}{8}}$

(c) Find the instantaneous rate of change of $f(x)$ at $x=6$.

$f'(6)=\dfrac{1}{2\sqrt{6-5}}=\boxed{\dfrac{1}{2}}$

(d) Find the equation of the tangent line at $x=14$.

$m=f'(14)=\dfrac{1}{2\sqrt{14-5}}=\dfrac{1}{6}$

point $=(14,f(14))=(14,6)$

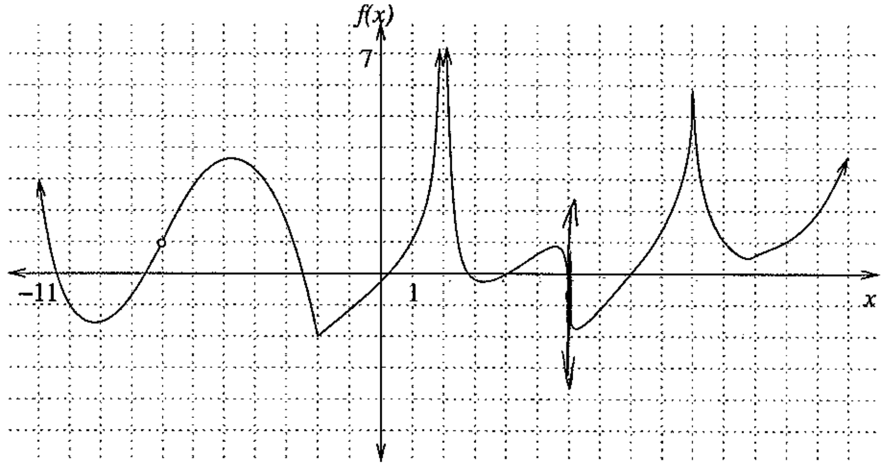
$f(14)=\sqrt{14-5}+3$

$=3+3=6$

$y-6=\frac{1}{6}(x-14)$

$\boxed{y=\frac{1}{6}x+\frac{11}{3}}$

3. The graph of a function f is shown below. Find all values of x for which $f(x)$ is not differentiable.



Not differentiable at	Because
$x=-7$	Discontinuous
$x=-2$	sharp point (corner)
$x=2$	Discontinuous
$x=6$	Vertical tangent
$x=10$	Sharp point (cusp)

4. Find the derivative of each of the following.

(a) $f(x)=7x+5$

$$\boxed{f'(x)=7}$$

(b) $g(x)=10x^5-7x^4+3x^2-4$

$$\boxed{g'(x)=50x^4-28x^3+6x}$$

(c) $h(x)=7\sqrt{x}-\frac{8}{x^{9.1}}+10\sqrt[3]{x^5}-\frac{11}{\sqrt[9]{x^4}}+2x^{-1.32}-6\pi^3$ (rewrite first)

$$h(x)=7x^{1/2}-8x^{-9.1}+10x^{5/3}-11x^{-4/9}+2x^{-1.32}-6\pi^3$$

$$\boxed{h'(x)=\frac{7}{2}x^{-1/2}+72.8x^{-10.1}+\frac{50}{3}x^{2/3}+\frac{44}{9}x^{-13/9}-2.64x^{-2.32}}$$

5. Find the value(s) of x where the tangent line to $f(x)=2x^5-30x^3+e^4$ is horizontal.

slope = 0

$$f'(x)=10x^4-90x^2$$

$$10x^4-90x^2=0$$

$$\boxed{x=0,3,-3}$$

$$10x^2(x^2-9)=0$$

$$10x^2(x-3)(x+3)=0$$

$10x^2=0$ $\quad x-3=0$ $\quad x+3=0$

Y_1

6. The total sales of a company t months from now are given by $S(t) = 0.004t^4 + 0.3t^3 + 2.7t^2 + 5t - 2$ thousand dollars.

   (a) Find a model for the rate of change of sales with respect to time.

   $S'(t) = 0.016t^3 + 0.9t^2 + 5.4t + 5$ thousand dollars per month, where t is the number of months from now.

   (b) Find and interpret $S(4)$ and $S'(4)$.

   $S(t) \to Y_1$
   $S'(t) \to Y_2$

   $S(4) = Y_1(4) = 81.424$
   $S'(4) = Y_2(4) = 42.024$

   Four months from now, the company's total sales will be \$81,424 and will be increasing at a rate of \$42,024 per month.

   (c) Use your answers in (b) to estimate the company's total sales 5 months from now and 6 months from now.

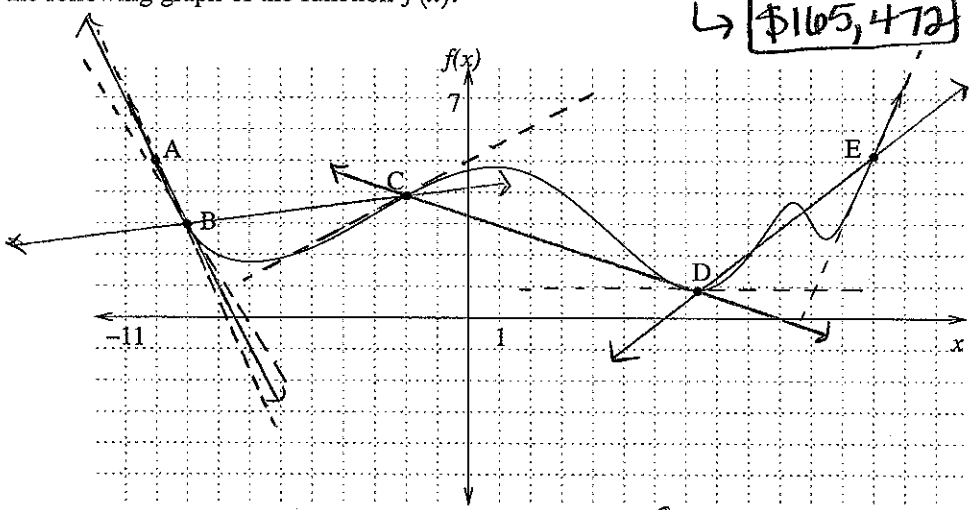
   $$S(5) \approx S(4) + S'(4) = 81.424 + 42.024 = 123.448 \to \boxed{\$123{,}448}$$

   $$S(6) \approx S(4) + 2S'(4) = 81.424 + 2(42.024) = 165.472 \to \boxed{\$165{,}472}$$

7. Consider the following graph of the function $f(x)$.

Solid lines are secant lines

dashed lines are tangent lines



   (a) Between which two consecutive labeled points is the $\boxed{\text{average rate of change}}$ positive? Negative?

   slope of secant line

   Positive between B and C and between D and E.
   Negative between A and B and between C and D

   (b) Between which two consecutive labeled points is the average rate of change largest? Smallest?

   steepest positive slope

   Largest between D and E
   Smallest between A and B

   (c) At which labeled point(s) is the instantaneous rate of change positive? Negative?

   slope of tangent line

   Positive at C and E

   Negative at A and B

   (d) At which labeled point(s) is the instantaneous rate of change zero?

   D

   (e) At which labeled point(s) is the instantaneous rate of change largest? Smallest?

   largest: E

   Smallest: A

8. Let $f(x) = \frac{1}{x}$.

(a) Use the limit definition of the derivative to find $f'(x)$.

① $f(x+h) = \frac{1}{x+h}$

② $f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x}$

$= \frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}$

$= \frac{x-(x+h)}{x(x+h)}$

$= \frac{-h}{x(x+h)}$

③ $\frac{f(x+h)-f(x)}{h} = \frac{\frac{-h}{x(x+h)}}{h}$

$= \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = -\frac{1}{x(x+h)}$

④ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{-1}{x(x+h)}$

$= \frac{-1}{x^2}$

so $\boxed{f'(x) = -\frac{1}{x^2}}$

(b) Confirm your answer using basic differentiation properties.

$f(x) = x^{-1}$

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$ ✓

(c) Are there any values of x for which $f'(x)$ does not exist?

$f'(x)$ DNE where $f(x)$ is discontinuous.

$f(x)$ is discontinuous at $x=0$, so $f'(x)$ DNE at $\boxed{x=0}$

9. If $h(x) = 7x + 3f(x) - 2g(x) + 8$, $f'(3) = 4$, and $g'(3) = -5$, find $h'(3)$.

$h(x) = 7x + 3f(x) - 2g(x) + 8$

$h'(x) = 7 + 3f'(x) - 2g'(x)$

$h'(3) = 7 + 3f'(3) - 2g'(3)$

$= 7 + 3(4) - 2(-5) = \boxed{29}$

10. Acme, Inc.'s market research department has determined the price-demand function for its graphing calculators to be $p = 201 - 0.03x$ dollars per calculator, where x is the number of graphing calculators demanded. Acme has a fixed cost of \$81,180 and a variable cost of \$36.90 per calculator.

(a) Find a model for revenue, and state its domain.

$R(x) = px = 201x - 0.03x^2$ dollars, where x is the number of graphing calculators sold, $\underbrace{0 \le x \le 6700}_{\text{Domain.}}$

Domain: $p \ge 0$ so $201 - 0.03x \ge 0$

$-0.03x \ge -201$

$x \le 6700$

(b) Find a model for marginal revenue.

$R'(x) = 201 - 0.06x$ dollars per calculator, where x is the number of graphing calculators sold.

(c) Find $R(3000)$ and $R'(3000)$ and interpret your answers.

$R(6000) = 201(3000) - 0.03(3000)^2 = \$333{,}000$

$R'(6000) = 201 - 0.06(3000) = \$21$ per calculator

When 3000 calculators are sold, Acme's revenue is \$333,000, and the revenue is increasing at a rate of \$21 per calculator.

(d) What is the approximate revenue generated by the sale of the 3,001st calculator?

a single calculator

Selling the 3001st calculator should generate approximately \$21 in revenue.

(e) Find the exact revenue generated by the sale of the 3,001st calculator.

$$\text{Rev. of 3001st} = R(3001) - R(3000)$$

$$= \left[201(3001) - 0.03(3001)^2\right] - \left[201(3000) - 0.03(3000)^2\right]$$

$$= 333020.97 - 333000$$

$$= \boxed{\$20.97}$$

(f) Find the cost and marginal cost of producing 2,500 calculators and use your answers to estimate the total cost for producing 2,501 calculators and producing 2,515 calculators.

$C(x) = 36.90x + 81180$ so $C(2500) = \$173{,}430$

$C'(x) = \$36.90$ per calc. so $C'(2500) = \$36.90$ per calculator

$C(2501) \approx C(2500) + C'(2500)$
$= 173430 + 36.90$
$= \boxed{\$173466.90}$

$C(2515) \approx C(2500) + 15C'(2500)$
$= 173430 + 15(36.90)$
$= \boxed{\$173983.50}$

(g) What is the average revenue when 6,000 calculators are sold?

$\overline{R}(x) = \frac{R(x)}{x} = \frac{201x - 0.03x^2}{x} = 201 - 0.03x$ dollars per calculator

$\overline{R}(6000) = 201 - 0.03(6000) = \boxed{\$21 \text{ per calculator}}$

(h) Find the average profit function.

$\overline{P}(x) = \frac{P(x)}{x} = \frac{(201x - 0.03x^2) - (36.9x + 81180)}{x} = \frac{-0.03x^2 + 164.1x - 81180}{x}$

$\boxed{\overline{P}(x) = -0.03x + 164.1 - 81180x^{-1} \text{ dollars per calculator}}$

(i) Find $\overline{P}'(3000)$ and interpret your answer.

$\overline{P}'(x) = -0.03 + 81180x^{-2}$ dollars/calc. per calculator

$\overline{P}'(3000) = -0.03 + \frac{81180}{3000^2} = -\$0.02$/calc per calculator

When 3000 calculators are produced and sold, the average profit per calculator is decreasing at a rate of \$0.02/calculator per calculator

(j) Find a model for marginal average revenue.

$\overline{R}'(x) = -\$0.03$/calculator per calculator,

where x is the number of calculators sold.

11. Annette plans to invest some money in an account paying 7.25% per year compounded continuously. How long will it take for her money to triple?

$A = Pe^{rt}$

$3P = Pe^{0.0725t}$

$3 = e^{0.0725t}$

$\ln 3 = 0.0725t$

$t = \frac{\ln 3}{0.0725} = \boxed{15.1533 \text{ years}}$

12. Bob has invested \$5,400 into a savings account paying 5.9% per year compounded continuously.

(a) How much interest will Bob earn during the next 5 years?

$A = Pe^{rt}$

$A = 5400e^{0.059(5)}$

$A = \$7,252.88$

Interest Earned

$7252.88 - 5400$

$= \boxed{\$1852.88}$

(b) How long will it take for Bob's account to reach \$8,000?

$8000 = 5400e^{0.059t}$

$\frac{40}{27} = e^{0.059t}$

$\ln\left(\frac{40}{27}\right) = 0.059t$

$t = \frac{\ln\left(\frac{40}{27}\right)}{0.059} = \boxed{6.6617 \text{ years}}$

13. A company's revenue can be modeled by $R(x) = 300x - 0.15x^2$ million dollars, where x is the number of items sold.

(a) Find the average revenue when 900 items are sold.

$\overline{R}(x) = \frac{300x - 0.15x^2}{x} = 300 - 0.15x$

$\overline{R}(900) = 300 - 0.15(900) = \boxed{\$165 \text{ million / item}}$

(b) Find the marginal revenue when 900 items are sold.

$R'(x) = 300 - 0.3x$

$R'(900) = 300 - 0.3(900) = \boxed{\$30 \text{ million per item}}$

(c) Find the marginal average revenue when 900 items are sold.

$\overline{R}'(x) = -0.15$

$\overline{R}'(900) = \boxed{-0.15 \text{ million dollars/item per item}}$

(d) Find the revenue when 900 items are sold.

$R(900) = 300(900) - 0.15(900)^2$

$= \boxed{\$148,500 \text{ million}}$

or \$148,500,000,000