Math 142 - Week in Review #6

1. Find the derivative of each of the following functions.

(a) $f(x) = 10e^x + 7x^4 + 3$

$$\boxed{f'(x) = 10e^x + 28x^3}$$

(b) $g(x) = 5 \cdot 3^x - \ln x - e^2$

$$\boxed{g'(x) = 5 \cdot 3^x \cdot \ln 3 - \frac{1}{x}}$$

(c) $h(x) = \ln(5x^2) + 3e^{2x} + 5\log_3 x$

Method 1: Chain Rule on $\ln(5x^2)$

$$h'(x) = \left(\frac{1}{5x^2}\right)(10x) + 3e^{2x}(2) + 5\left(\frac{1}{x}\right)\left(\frac{1}{\ln 3}\right)$$

$$\boxed{h'(x) = \frac{2}{x} + 6e^{2x} + \frac{5}{x \cdot \ln 3}}$$

Method 2: Properties of Logs

$$h(x) = \ln 5 + \ln x^2 + 3e^{2x} + 5\log_3 x$$

$$h(x) = \ln 5 + 2\ln x + 3e^{2x} + 5\log_3 x$$

$$h'(x) = \frac{2}{x} + 3e^{2x}(2) + 5\left(\frac{1}{x}\right)\left(\frac{1}{\ln 3}\right)$$

$$\boxed{h'(x) = \frac{2}{x} + 6e^{2x} + \frac{5}{x\ln 3}}$$

(d) $f(t) = \left(6t^4 - \frac{7}{t^5}\right)^{10} = (6t^4 - 7t^{-5})^{10}$

$$\boxed{f'(t) = 10\left(6t^4 - 7t^{-5}\right)^9\left(24t^3 + 35t^{-6}\right)}$$

(e) $m(x) = 5\left(7.2^{x^4-7x^2}\right)$

$$m'(x) = 5(\ln 7.2)\left(7.2^{x^4-7x^2}\right)(4x^3-14x)$$

(f) $h(t) = \ln(14t^2 - 3t^9)$

$$h'(t) = \frac{1}{14t^2-3t^9}(28t - 27t^8) = \frac{28t-27t^8}{14t^2-3t^9}$$

2. The value of a boat t months after it was purchased can be modeled by $V(t) = 37,500\,(0.97^t)$ dollars.

(a) What is the value of the boat 1 year after it is purchased?

$$V(12) = 37500(0.97^{12}) = \boxed{\$26,019.09}$$

(b) What is the rate of depreciation 1 year after it is purchased?

$V'(t) = 37500(\ln 0.97)(0.97^t)$ dollars per month

$$V'(12) = 37500(\ln 0.97)(0.97^{12})$$

$$= \boxed{-\$792.52 \text{ per month}}$$

One year after purchase, the value of the boat is depreciating (decreasing) at a rate of \$792.52 per month

(c) Use your answers to (a) and (b) to estimate the value of the boat 15 months after it was purchased.

$$V(15) \approx V(12) + 3 \cdot V'(12)$$

$$= 26,019.09 + 3(-792.52)$$

$$= \boxed{\$23,641.53}$$

3. Find the derivative of each of the following functions.

(a) $m(x) = e^{4x}(x^2 - \ln 4x)$

$$m'(x) = 4e^{4x}(x^2 - \ln 4x) + e^{4x}\left(2x - \frac{1}{\cancel{4}x}(\cancel{4})\right)$$

$$\boxed{m'(x) = 4e^{4x}(x^2 - \ln 4x) + e^{4x}\left(2x - \frac{1}{x}\right)}$$

(b) $h(t) = \dfrac{4t^{-2} - 7t}{2^t + 1}$

$$h'(t) = \frac{(2^t+1)(-8t^{-3} - 7) - (4t^{-2} - 7t)((\ln 2)\cdot 2^t)}{(2^t+1)^2}$$

(c) $f(x) = (\log_7 x)^5 \left(4x - \sqrt{5-x}\right)$ $\quad (5-x)^{1/2}$

$$f'(x) = 5(\log_7 x)^4 \cdot \left(\left(\frac{1}{x}\right)\left(\frac{1}{\ln 7}\right)\right)\left(4x - (5-x)^{1/2}\right) + (\log_7 x)^5\left(4 - \frac{1}{2}(5-x)^{-\frac{1}{2}}(-1)\right)$$

(d) $g(x) = \dfrac{5^x - \ln x}{\sqrt[3]{(e^x + x)^2}} = \dfrac{5^x - \ln x}{(e^x + x)^{2/3}}$

$$g'(x) = \frac{(e^x+x)^{2/3}\left((\ln 5)\cdot 5^x - \frac{1}{x}\right) - (5^x - \ln x)\left(\frac{2}{3}(e^x+x)^{-1/3}(e^x+1)\right)}{\left[(e^x+x)^{2/3}\right]^2}$$

(so denominator is $(e^x + x)^{\frac{4}{3}}$)

4. Let f and g be functions such that

$$f(5) = 10 \qquad f'(5) = 8 \qquad f(9) = 30 \qquad f'(9) = -5$$
$$g(5) = 9 \qquad g'(5) = -4.6 \qquad g(10) = 13 \qquad g'(10) = 2.1$$

(a) Find $h'(5)$ if $h(x) = \dfrac{f(g(x))}{x+1}$.

$$h'(x) = \frac{(x+1) f'(g(x))g'(x) - f(g(x))(1)}{(x+1)^2}$$

$$h'(5) = \frac{(5+1) f'(g(5))g'(5) - f(g(5))}{(5+1)^2} = \frac{6f'(9)\cdot(-4.6) - f(9)}{36}$$

$$= \frac{6(-5)(-4.6) - 30}{36} = \boxed{3}$$

(b) Find $m'(5)$ if $m(x) = -3g(x)\sqrt{f(x)}$.

$$m'(x) = -3g'(x)\,[f(x)]^{\frac{1}{2}} + (-3g(x))\left[\tfrac{1}{2}[f(x)]^{-1/2} f'(x)\right]$$

$$m'(5) = -3g'(5)[f(5)]^{1/2} - \tfrac{3}{2}g(5)\cdot[f(5)]^{-1/2} f'(5)$$

$$= -3(-4.6)(10)^{1/2} - \tfrac{3}{2}(9)(10)^{-1/2}(8)$$

$$\approx \boxed{9.4868}$$

5. Find the derivative of each of the following functions.

(a) $f(x) = \sqrt{\dfrac{8}{\sqrt[4]{x^5}} + 5e^{-x} + 13} = \left(8x^{-\frac{5}{4}} + 5e^{-x} + 13\right)^{\frac{1}{2}}$

$$f'(x) = \tfrac{1}{2}\left(8x^{-\frac{5}{4}} + 5e^{-x} + 13\right)^{-\frac{1}{2}}\left(-10x^{-\frac{9}{4}} + 5e^{-x}(-1)\right)$$

(b) $g(x) = [\log_3(\log_7(10 - x^2))]^5$

$$g'(x) = 5\left[\log_3(\log_7(10-x^2))\right]^4 \cdot \frac{d}{dx}\left[\log_3(\log_7(10-x^2))\right]$$

$$g'(x) = 5\left[\log_3(\log_7(10-x^2))\right]^4 \cdot \left(\frac{1}{\ln 3}\right)\left(\frac{1}{\log_7(10-x^2)}\right)\cdot \frac{d}{dx}\left[\log_7(10-x^2)\right]$$

$$\boxed{g'(x) = 5\left[\log_3(\log_7(10-x^2))\right]^4 \cdot \left(\frac{1}{\ln 3}\right)\left(\frac{1}{\log_7(10-x^2)}\right)\left(\frac{1}{\ln 7}\right)\left(\frac{1}{10-x^2}\right)(-2x)}$$

6. Find the values of x where the $\boxed{\text{tangent line is horizontal}}$ for each of the following functions.

$\leftarrow f'(x) = 0$

(a) $f(x) = \frac{3x}{(7x+1)^2}$

$$f'(x) = \frac{(7x+1)^2(3) - 3x[2(7x+1)(7)]}{(7x+1)^4}$$

$$f'(x) = \frac{3(7x+1)[(7x+1) - 14x]}{(7x+1)^4} = \frac{3\cancel{(7x+1)}(-7x+1)}{(7x+1)^{\cancel{4}\,3}} = \frac{3(-7x+1)}{(7x+1)^3} \stackrel{\downarrow}{=} 0$$

$3(-7x+1) = 0$

$-7x+1 = 0$

$-7x = -1$

$\boxed{x = \frac{1}{7}}$

(b) $g(x) = x^4(x-5)^3$

$$g'(x) = 4x^3(x-5)^3 + x^4[3(x-5)^2(1)] = 0$$

$$4x^3(x-5)^3 + 3x^4(x-5)^2 = 0$$

$$x^3(x-5)^2[4(x-5) + 3x] = 0$$

$$x^3(x-5)^2(7x-20) = 0$$

$x^3 = 0$ $\quad$ $x-5 = 0$

$x = 0$ $\quad$ $x = 5$

$7x - 20 = 0$

$7x = 20$

$x = \frac{20}{7}$

$\boxed{x = 0, \frac{20}{7}, \text{ and } 5}$

7. Find the equation of the tangent line to the graph of $f(x) = \sqrt{\ln x} + e^{\sqrt{x}}$ at $x = 4$.

$f(x) = (\ln x)^{1/2} + e^{x^{1/2}}$

① $m = f'(4)$

$$f'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right) + e^{x^{\frac{1}{2}}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$f'(4) = \frac{1}{2}(\ln 4)^{-1/2}\left(\frac{1}{4}\right) + e^{\sqrt{4}}\left(\frac{1}{2}(4)^{-1/2}\right)$$

$$f'(4) = \frac{1}{8}\left(\frac{1}{\sqrt{\ln 4}}\right) + e^2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$f'(4) \approx 1.9534$$

② $(4, f(4)) = (4, 8.5665)$

$f(4) = \sqrt{\ln 4} + e^{\sqrt{4}} \approx 8.5665$

③ $y - 8.5665 = 1.9534(x - 4)$

$\boxed{y = 1.9534x + 0.7529}$

8. Use properties of logarithms to prove that $\frac{dy}{dx} = 0$ for $y = \ln 7e^{2x} - \ln(4e^x)^2$.

$$y = \ln 7 + \ln e^{2x} - 2\ln(4e^x)$$

$$= \ln 7 + 2x - 2(\ln 4 + \ln e^x)$$

$$= \ln 7 + 2x - 2\ln 4 - 2\ln e^x$$

$$= \ln 7 + 2x - 2\ln 4 - 2x$$

$= \ln 7 - 2\ln 4$ $\leftarrow$ this is a constant, so $\frac{dy}{dx} = 0$.

9. The number of mice on an uninhabited island in the Pacific Ocean can be modeled by $P(t) = \dfrac{6{,}000}{1+290e^{-1.8t}}$ mice, where t is the number of years since the end of 1982.

   (a) Find $P(2)$ and $P'(2)$ and interpret your answers.

   $\boxed{\text{At the end of 1984, there were 672 mice on the island, and the number of mice was increasing at a rate of 1075 mice per year.}}$

$$P(2) = \frac{6000}{1+290e^{-1.8(2)}} = 672.3533 \approx 672 \text{ mice}$$

$$P'(t) = \frac{0 - 6000(290e^{-1.8t}(-1.8))}{(1+290e^{-1.8t})^2} = \frac{3{,}132{,}000e^{-1.8t}}{(1+290e^{-1.8t})^2} \Bigg\} \text{store in } Y_1$$

$$P'(2) = Y_1(2) = 1074.62 \approx 1075 \text{ mice per year}$$

   (b) Use your answer in (a) to approximate the number of mice that will be added to the island's population during 1985.

   In 1985, approximately 1075 mice will be added to the island's mouse population.

   (c) Use your answer in (a) to approximate the island's mouse population at the end of 1987.

$$P(5) \approx P(2) + 3P'(2)$$
$$= 672 + 3(1075) = \boxed{3897 \text{ mice}}$$

10. Find the derivative of each of the following functions.

   (a) $m(x) = e^{e^x}\left(\dfrac{x\sqrt[3]{x^8}}{\sqrt[6]{x^4}}\right)$ $= e^{e^x}\left(\dfrac{x \cdot x^{\frac{8}{3}}}{x^{\frac{4}{6}}}\right) = e^{e^x}\left(x^{1+\frac{8}{3}-\frac{2}{3}}\right) = e^{e^x}(x^3)$

$$m'(x) = (e^{e^x})(e^x)(x^3) + e^{e^x}(3x^2)$$

   (b) $k(x) = \dfrac{8^x}{x\log_5 x}$

$$k'(x) = \frac{(x\log_5 x)(8^x \ln 8) - 8^x\left(1\cdot\log_5 x + x\left(\frac{1}{x}\right)\left(\frac{1}{\ln 5}\right)\right)}{(x\log_5 x)^2}$$