Math 166 - Week in Review #7

Section 4.3 - Gauss Elimination for Systems of Linear Equations

- When a system of linear equations has only two variables, each equation represents a line and "solving the system" means finding all points the lines have in common.
- For any system of n linear equations in n variables, there are only 3 possibilities for the solution: (1) a unique solution, (2) infinitely many solutions, or (3) no solution.
- If a system of equations has infinitely many solutions, you MUST give the parametric solution for the system.
- Gauss Elimination - The goal of Gauss Elimination is to use elementary equation (or row) operations on a given system of equations to obtain an equivalent system that is in triangular form like the following:

$$\begin{array}{rcrcrcl} x & + & ay & + & bz & = & c \\ & & y & + & dz & = & e \\ & & & & z & = & f \end{array}$$

  Then we can use back-substitution to solve for x, y, and z.

- Elementary Equation Operations
  1. Two equations can be interchanged, $E_i \leftrightarrow E_j$.
  2. An equation may be multiplied by a non-zero constant, $kE_i \rightarrow E_i$.
  3. A multiple of one equation may be added to another equation, $E_i + kE_j \rightarrow E_i$.
- Steps for Gauss Elimination - To solve a system of equations in the n unknowns $x_1, x_2, \ldots, x_n$, use elementary equation operations for each of the following:
  1. Transform the system of equations so that the coefficient of x_1 in the first equation is 1.
  2. Eliminate the x_1's from all equations below the first equation.
  3. Tranform the new system so that the coefficient of x_2 in the second equation is 1.
  4. Eliminate the x_2's from all equations below the second equation.
  5. Continue in a like manner until you obtain a system of equations that is in triangular form.
  6. Use back-substitution to find the values of all the variables.
- Gauss Elimination Using an Augmented Matrix - For larger systems, it is convenient to first write them in augmented matrix form and then apply Gauss elimination with elementary row operations to solve the system. When using an augmented matrix, the goal of Gauss elimination is to use the elementary row operations to transform the matrix into **echelon form**.
- Elementary Row Operations
  1. Interchange the row i with the row j ($R_i \leftrightarrow R_j$).
  2. Multiply each element of row i by a nonzero constant k ($kR_i \rightarrow R_i$).
  3. Replace each element in row i with the corresponding element in row i plus k times the corresponding element in row j ($R_i + kR_j \rightarrow R_i$).

- Echelon Form - A matrix is in **echelon form** if
  1. The first nonzero element in any row is 1, called the leading one.
  2. The column containing the leading one has all elements below the leading one equal to 0.
  3. The leading one in any row is to the left of the leading one in a lower row.
  4. Any row consisting of all zeros must be below any row with at least one nonzero element.
- Gauss-Jordan Elimination Method - an extension of the Gauss elimination method in which row operations are used to transform the augmented matrix into a simpler form called *reduced row echelon form*

Section 4.4 - Systems of Linear Equations with Non-unique Solutions

- When a system of equations has infinitely many solutions, we must give the **parametric solution.**
- Using RREF to Solve Systems of Equations
  STEP 1: Check the final matrix to see if there is no solution. (If the system has no solution, state so and stop here. Otherwise, go on to Step 2.)
  STEP 2: Circle the leading 1's.
  a) If each variable has a leading 1 in its column, then there is a unique solution.
  b) Otherwise, there are (potentially) multiple solutions and each variable not having a leading one in its column is a parameter. **NOTE: If the variables in the system of equations represent quantities or units of some items, then you must consider whether it is necessary to put restrictions on the parameters (and do so if it is necessary).**
- Overdetermined systems have more equations than unknowns. These systems can have a unique solution, infinitely many solutions, or no solution.
- Underdetermined systems have fewer equations than unknowns. These systems can only have infinitely many solutions or no solution.

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1. Solve the system of equations $\begin{aligned} 2\left(x - \tfrac{3}{2}y \right.&= \left.-2\right) \\ 6\left(\tfrac{2}{3}x - \tfrac{5}{6}y \right.&= \left.\tfrac{7}{3}\right) \end{aligned}$ using elementary equation operations.

$2E_1 \to E_1$
$6E_2 \to E_2$

$\left.\begin{aligned} 2x - 3y &= -4 \\ 4x - 5y &= 14 \end{aligned}\right\}$ New System

$-2(2x - 3y = -4)$

$$\begin{aligned} -4x + 6y &= 8 \\ +4x - 5y &= 14 \\ \hline y &= 22 \end{aligned}$$

$x - \frac{3}{2}(22) = -2$

$x - 33 = -2$

$x = 31$

$\boxed{(31, 22)}$

2. Solve the system of equations $\begin{aligned} 3x-6y &= 18 \\ -2x+4y &= -12 \end{aligned}$ using Gauss elimination with an augmented matrix.

$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 3 & -6 & 18 \\ -2 & 4 & -12 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & 6 \\ -2 & 4 & -12 \end{array}\right] \xrightarrow{R_2+2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y=t$, where t is any real number.

$x-2y=6$

$x=2y+6$

$x=2t+6$

$\boxed{(x,y)=(2t+6,t)}$

Infinitely many solutions.

$x-2y=6$

$0x+0y=0 \leftarrow 0=0$ true-

count leading ones.

3. Give three particular solutions to the system solved in the previous problem.

When $t=0$, $(x,y)=(2(0)+6,0)=\boxed{(6,0)}$

when $t=1$, $(x,y)=(2(1)+6,1)=\boxed{(8,1)}$

when $t=-2$, $(x,y)=(2(-2)+6,-2)=\boxed{(2,-2)}$

4. A system of equations with infinitely many solutions can be represented by the parametric solution $(x_1,x_2,x_3,x_4)=(4-t,s,-2s+6t,t)$ where s and t are any real numbers.

(a) Which of the following could be particular solutions of the system?

A) $(3,3,0,1)$ $s=3$ $t=1$
$4-1=3$ ✓
$-2(3)+6(1)=0$ ✓

~~B)~~ $(4,3,-2,1)$ $s=3$ $t=1$
$4-1=3 \leftarrow$ doesn't match

C) $(-3,-1,44,7)$ $s=-1$ $t=7$
$4-7=-3$ ✓
$-2(-1)+6(7)$
$=2+42$
$=44$ ✓

~~D)~~ $(9,1,-3,-5)$ $s=1$ $t=-5$
$4-(-5)=9$ ✓
$-2(1)+6(-5)$
$=-2-30$
$=-32 \leftarrow$ doesn't match

~~E)~~ $(3,-4,2,1)$ $s=-4$ $t=1$
$4-1=3$ ✓
$-2(-4)+6(1)$
$=8+6$
$=14$ doesn't match

(b) Give two other particular solutions to this system.

① When $s=0$ and $t=0$,
$(x_1,x_2,x_3,x_4)=(4-0,0,-2(0)+6(0),0)=\boxed{(4,0,0,0)}$

② When $s=0$ and $t=1$,
$(x_1,x_2,x_3,x_4)=(4-1,0,-2(0)+6(1),1)=\boxed{(3,0,6,1)}$

5. Solve the system of equations $\begin{aligned} 6x - y - 2z &= 47 \\ 11x - 2y - 3z &= 82 \\ 5x - y - 2z &= 42 \end{aligned}$ using Gauss elimination with an augmented matrix.

make this a one

$$\begin{array}{ccc|c} x & y & z & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & -1 & -2 & 47 \\ 11 & -2 & -3 & 82 \\ 5 & -1 & -2 & 42 \end{array}\right] \xrightarrow{R_1 - R_3 \to R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 11 & -2 & -3 & 82 \\ 5 & -1 & -2 & 42 \end{array}\right]$$

make this a zero

$$\xrightarrow{R_2 - 11R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -2 & -3 & 27 \\ 5 & -1 & -2 & 42 \end{array}\right] \xrightarrow{R_3 - 5R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & -2 & -3 & 27 \\ 0 & -1 & -2 & 17 \end{array}\right]$$

make this a one

make this a one

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & \frac{3}{2} & -\frac{27}{2} \\ 0 & -1 & -2 & 17 \end{array}\right] \xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & \frac{3}{2} & -\frac{27}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{7}{2} \end{array}\right]$$

make this a zero

make this a one

$$\xrightarrow{-2R_3 \to R_3} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & \frac{3}{2} & -\frac{27}{2} \\ 0 & 0 & 1 & -7 \end{array}\right] \end{array} \xrightarrow[\text{Gauss-Jordan or elimination}]{R_2 - \frac{3}{2}R_3 \to R_2} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -7 \end{array}\right] \end{array} \begin{array}{l} x=5 \\ y=-3 \\ z=-7 \end{array}$$

Gauss elimination

$x \qquad = 5$

$y + \frac{3}{2}z = -\frac{27}{2} \to y + \frac{3}{2}(-7) = -\frac{27}{2}$

$z = -7 \qquad\qquad y = -3$

$(x, y, z) = (5, -3, -7)$

$$\boxed{(x, y, z) = (5, -3, -7)}$$

use rref in calculator

6. Solve the following systems of equations using $\boxed{\text{any method.}}$ If there are infinitely many solutions, state so and give the parametric solution. If there is no solution, state so.

(a) $\begin{array}{rcrcrcr} 3x & + & 4y & - & z & = & -8 \\ 2x & + & 5y & + & z & = & -3 \end{array}$

$$\left[\begin{array}{ccc|c} 3 & 4 & -1 & -8 \\ 2 & 5 & 1 & -3 \end{array}\right] \xrightarrow{rref} \begin{array}{cccc} x & y & z & \\ \left[\begin{array}{ccc|c} 1 & 0 & -9/7 & -4 \\ 0 & 1 & 5/7 & 1 \end{array}\right] \end{array}$$ only two leading 1's

Let $z=t$, where t is any real number.

$$x - \frac{9}{7}z = -4 \qquad\qquad y + \frac{5}{7}z = 1$$

$$x = \frac{9}{7}z - 4 \qquad\qquad y = -\frac{5}{7}z + 1$$

$$x = \frac{9}{7}t - 4 \qquad\qquad y = -\frac{5}{7}t + 1$$

$$\boxed{(x,y,z) = \left(\frac{9}{7}t - 4, -\frac{5}{7}t + 1, t\right)}$$

Infinitely many solutions.

(b) $\begin{array}{rcrcr} x & + & 2y & = & 3 \\ 3x & + & 3y & = & 7 \\ 2x & + & y & = & 4 \end{array}$

$$\begin{array}{c} x \quad y \\ \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 3 & 7 \\ 2 & 1 & 4 \end{array}\right] \end{array} \xrightarrow{rref} \begin{array}{c} x \quad y \\ \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \to x = \frac{5}{3} \\ \to y = \frac{2}{3} \\ \to 0x + 0y = 0 \end{array}$$

$x = \frac{5}{3}$, $y = \frac{2}{3}$ $\to$ $\boxed{(x,y) = \left(\frac{5}{3}, \frac{2}{3}\right)}$ unique solution.

$0x+0y=0$ so $0=0$ True (go count leading 1's)

(c) $\begin{array}{rcrcr} 7x & - & 14y & = & 14 \\ -4x & + & 8y & = & -8 \\ 3x & - & 6y & = & -9 \end{array}$

$$\left[\begin{array}{cc|c} 7 & -14 & 14 \\ -4 & 8 & -8 \\ 3 & -6 & -9 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$\leftarrow 0x + 0y = 1 \rightarrow 0 = 1$ False!

$\leftarrow 0x + 0y = 0$ True ✓

$\boxed{\text{No solution}}$

(d) $\begin{array}{rcrcrcrcr} -x & + & 9y & - & 3z & + & 2w & = & 3 \\ -8x & + & 72y & - & 23z & + & 22w & = & 33 \\ 2x & - & 18y & + & 6z & - & 4w & = & -6 \end{array}$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left[\begin{array}{cccc|c} -1 & 9 & -3 & 2 & 3 \\ -8 & 72 & -23 & 22 & 33 \\ 2 & -18 & 6 & -4 & -6 \end{array}\right] \xrightarrow{rref} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & -9 & 0 & -20 & -30 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\leftarrow 0=0$ True, count leading 1's.

Let $y = s$, where s is any real number.
Let $w = t$, " " t " " " " " "

$x - 9y - 20z = -30$
$x = 9y + 20z - 30$
$x = 9s + 20t - 30$

$z + 6w = 9$
$z = -6w + 9$
$z = -6t + 9$

$$\boxed{(x, y, z, w) = (9s + 20t - 30,\ s,\ -6t + 9,\ t)}$$

Infinitely many solutions.

7. For the next five word problems do the following:

   I) Define the variables that are used in setting up the system of equations.

   II) Set up the system of equations that represents this problem.

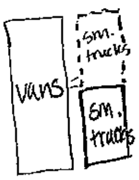
   III) Solve for the solution.

   IV) If the solution is parametric, then tell what restrictions should be placed on the parameter(s). Also give three specic solutions.

   (a) (#49, pg. 74 of *Finite Mathematics* by Lial, et. al.) The U-Drive Rent-A-Truck Co. plans to spend $6 million on 200 new vehicles. Each van will cost $20,000, each small truck $30,000, and each large truck $50,000. Past experience shows that they need twice as many vans as small trucks. How many of each kind of vehicle can they buy?

   Let x = the number of vans they can buy.
   Let y = the number of small trucks they can buy.
   Let z = the number of large trucks they can buy.

$$x + y + z = 200$$
$$20000x + 30000y + 50000z = 6,000,000$$
$$x = 2y$$



$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 20000 & 30000 & 50000 & 6000000 \\ 1 & -2 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{rref} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 50 \end{array}\right] \end{array} \begin{array}{l} x=100 \\ y=50 \\ z=50 \end{array}$$

They can buy 100 vans, 50 small trucks, and 50 large trucks.

   (b) A cashier has a total of 96 bills in his register in one-, five-, and ten-dollar denominations. If he has three times as many fives as ones, and if the number of ones and fives combined is half of the number of tens he has, how many bills of each denomination does he have in his register?

   Let x = the number of one-dollar bills in the register.
   Let y = the number of five-dollar bills in the register.
   Let z = the number of ten-dollar bills in the register.

$$\begin{array}{rcl} x + y + z = 96 & & x + y + z = 96 \\ y = 3x & \Rightarrow & -3x + y = 0 \\ x + y = \frac{1}{2}z & & x + y - \frac{1}{2}z = 0 \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 96 \\ -3 & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 \end{array}\right] \end{array} \xrightarrow{rref} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 24 \\ 0 & 0 & 1 & 64 \end{array}\right] \end{array} \begin{array}{l} x=8 \\ y=24 \\ z=64 \end{array}$$

He has 8 one-dollar bills, 24 five-dollar bills, and 64 ten-dollar bills in his register.

(c) Random, Inc. makes picture collages in three sizes. A small collage requires 30 minutes of cutting time and 36 minutes of pasting time. A medium collage requires 60 minutes of cutting time and 54 minutes for pasting. A large collage requires 90 minutes for cutting and 72 minutes for pasting. There are 380 labor hours available for cutting and 330 labor hours available for pasting each week. If the company wants to run at full capacity and wants to make twice as many small collages as medium collages, how many collages of each size should be made each week?

small collages = med. + med. collages

Let x = the number of small collages that should be made each week.
Let y = " " " medium " " – – – – – – .
Let z = " " " large " " – – – – .

	Small	Medium	Large	Amt Avail
Cutting time	30min	60min	90min	380 hrs * 60 = 22800 min
Pasting time	36min	54min	72min	330 hrs * 60 = 19800 min

$$30x + 60y + 90z = 22800$$
$$36x + 54y + 72z = 19800$$
$$x = 2y$$

$$\Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 30 & 60 & 90 & 22800 \\ 36 & 54 & 72 & 19800 \\ 1 & -2 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{rref} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 104 \\ 0 & 1 & 0 & 52 \\ 0 & 0 & 1 & 184 \end{array}\right] \end{array} \begin{array}{l} x=104 \\ y=52 \\ z=184 \end{array}$$

They should make 104 small collages, 52 medium collages, and 184 large collages each week.

(d) The management of a private investment club has a fund of $300,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that the management is considering have been classified into three categories: high-risk, medium-risk, and low-risk. Management estimates that high-risk stocks will have a rate of return of 16 percent per year; medium-risk stocks, 10 percent per year; and low-risk stocks, 4 percent per year. The investment in medium-risk stocks is to be twice the investment in stocks of the other two categories combined. If the investment goal is to have an average rate of return of 11 percent per year on the total investment, determine how much the club should invest in each type of stock.

Let x = the amount of money invested in high-risk stocks (in dollars).
Let y = the amount of money invested in medium-risk stocks " " .
Let z = " " – – – – – – – low-risk – – – – .

$$x + y + z = 300000 \qquad\qquad x+y+z = 300{,}000$$
$$y = 2(x+z) \qquad \Rightarrow \qquad -2x+y-2z = 0$$
$$0.16x + 0.1y + 0.04z = 0.11(300000) \qquad 0.16x+0.1y+0.04z = 33{,}000$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 300000 \\ -2 & 1 & -2 & 0 \\ 0.16 & 0.1 & 0.04 & 33000 \end{array}\right] \end{array} \xrightarrow{rref} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 75{,}000 \\ 0 & 1 & 0 & 200{,}000 \\ 0 & 0 & 1 & 25{,}000 \end{array}\right] \end{array} \begin{array}{l} x=75000 \\ y=200000 \\ z=25000 \end{array}$$

The club should invest $75,000 in high-risk stocks, $200,000 in medium-risk stocks, and $ 25,000 in low-risk stocks.

(e) A convenience store sold 23 sodas one summer afternoon in 12-, 16-, and 20-ounce cups (small, medium, and large). The total volume of soda sold was 376 ounces, and the total revenue was $48. If the prices for small, medium, and large sodas are $1, $2, and $3 respectively, how many of each size did the store sell that day? (pg. 70-72, *Finite Mathematics* by Lial, et. al.)

Let x = the number of small sodas sold.
Let y = the number of medium sodas sold.
Let z = the number of large sodas sold.

$$\begin{aligned} x+y+z&=23\\ 12x+16y+20z&=376\\ x+2y+3z&=48 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 23\\ 12 & 16 & 20 & 376\\ 1 & 2 & 3 & 48 \end{array}\right] \xrightarrow{rref} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & 1 & 2 & 25\\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

0=0 True (count leading 1's)

Let $z=t$, where t is an integer and $0 \le t \le 23$ ⊛ (this may need to be adjusted.)

Solve for x

$x-z=-2$

$x=z-2$

$x=t-2$

$x \ge 0$ so $t-2 \ge 0$

$t \ge 2$

restriction

Solve for y

$y+2z=25$

$y=-2z+25$

$y=-2t+25$

$y \ge 0$ so $-2t+25 \ge 0$

$-2t \ge -25$

$t \le 12.5$

restriction

Parametric Solution:

$$\boxed{(x,y,z)=(t-2,\,-2t+25,\,t) \text{ where } t=2,3,4,\ldots,12.}$$ (11 solutions)

Give 3 particular solutions:

When $t=2$, $(x,y,z)=(2-2,-2(2)+25,2)=(0,21,2)$ (0 small, 21 med., 2 lg sodas)

When $t=3$, $(x,y,z)=(3-2,-2(3)+25,3)=(1,19,3)$ (1 small, 19 med., 3 lg sodas)

When $t=12$, $(x,y,z)=(12-2,-2(12)+25,12)=(10,1,12)$ (10 small, 1 med, 12 lg sodas)