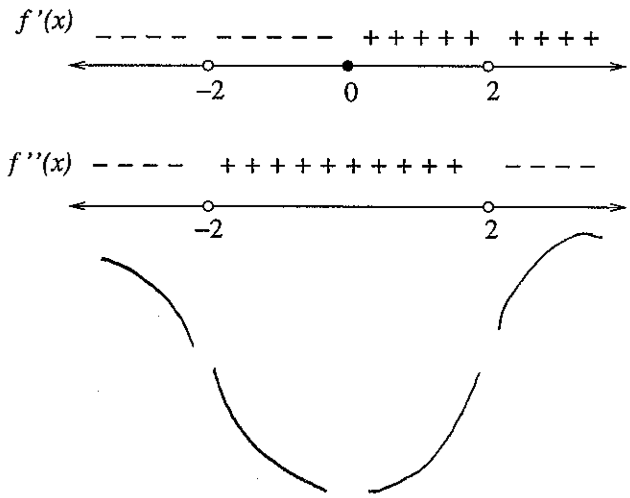
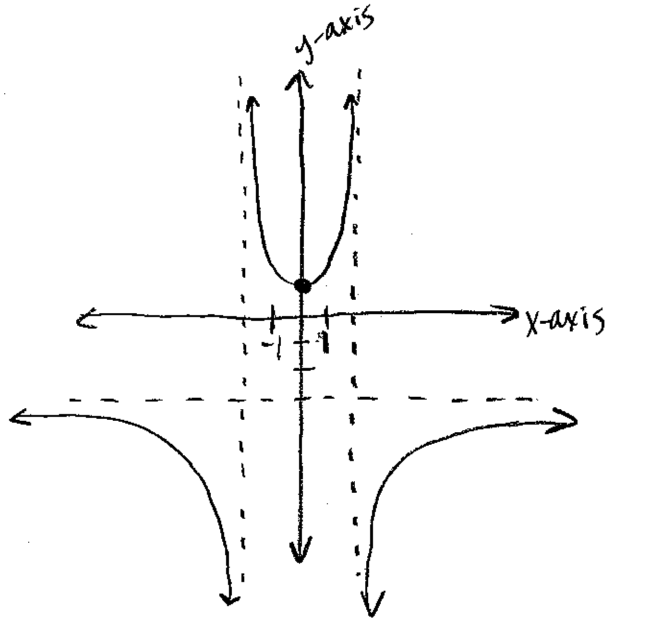
Math 142 - Week in Review #8

1. Sketch the graph of a function f that satisfies the following:

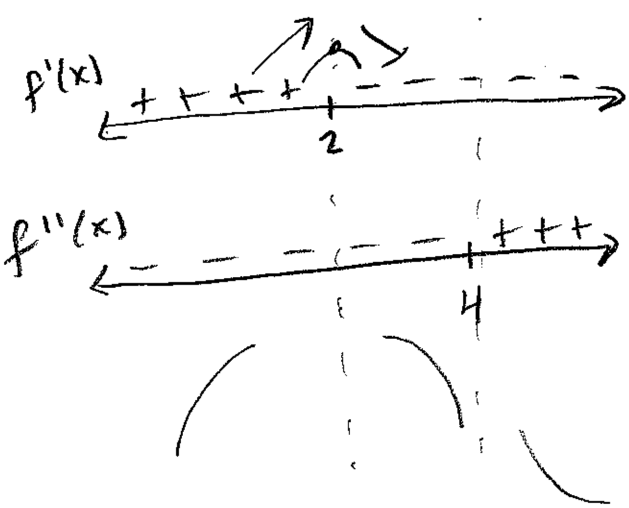
- Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
- Vertical asymptotes: $x=-2$ and $x=2$
- Horizontal asymptote: $y=-3$
- x-intercepts: none; y-intercept: $(0,1)$

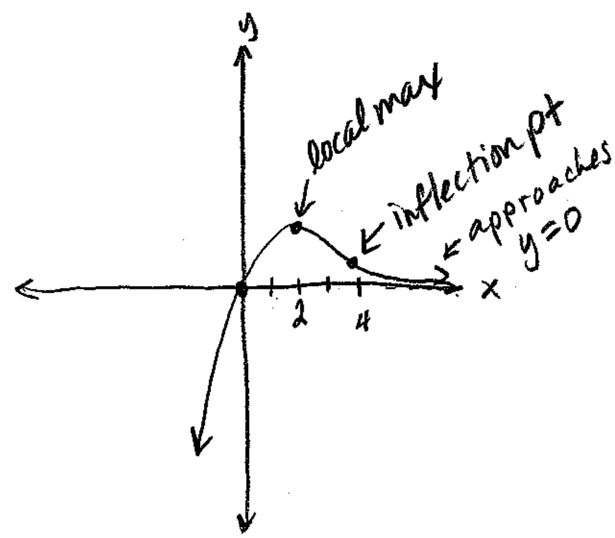




2. Sketch the graph of a function f that satisfies the following:

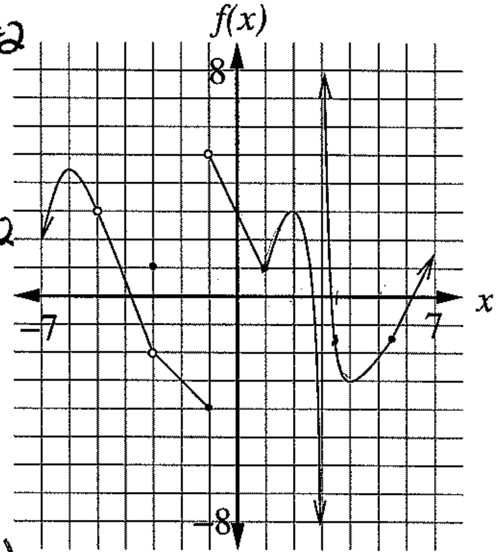
- Domain: All real numbers
- x and y-intercept: $(0,0)$
- Vertical asymptotes: none
- $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = 0$
- $f'(x) > 0$ on $(-\infty, 2)$
- $f'(x) < 0$ on $(2,\infty)$
- $f''(x) > 0$ on $(4,\infty)$
- $f''(x) < 0$ on $(-\infty, 4)$





x-coords ↓ ↓ y-coords

3. Use the given graph to find the absolute extrema (locations and values) of $f(x)$ on each of the intervals below.



(a) [0,2]

abs. max value is 3 and occurs at $x=0$ and $x=2$

abs. min value is 1 and occurs at $x=1$

(b) [0,3)

abs. max value is 3 and occurs at $x=0$ and $x=2$

no absolute minimum value on [0,3).

(c) [1,5]

no absolute maximum value on [1,5]

no absolute minimum value on [1,5]

(d) ~~(2,5,∞)~~ $(3.5, \infty)$

No absolute maximum value on $(3.5,\infty)$

absolute minimum value is -3 and occurs at $x=4$.

(e) $(-5,-3)$

no absolute maximum value

no absolute minimum value

4. Find the absolute extrema (locations and values) of $f(x) = x^2 e^{-0.4x}$ on

(a) $[-2,8]$

$$f'(x) = 2x e^{-0.4x} + x^2 e^{-0.4x}(-0.4)$$

$$= x e^{-0.4x}(2 - 0.4x) = 0$$

$x = 0$ $\quad e^{-0.4x} = 0$ no soln $\quad 2 - 0.4x = 0$, $x = 5$

x	$f(x) = x^2e^{-0.4x}$
-2	8.9022
0	0
5	3.3834
8	2.6088

The absolute maximum value on $[-2,8]$ is 8.9022 and occurs at $x=-2$, and the absolute minimum value is 0 and occurs at $x=0$.

(b) $[3,12]$

x	$f(x) = x^2e^{-0.4x}$
3	2.7107
5	3.3834
12	1.1851

The absolute maximum value on $[3,12]$ is 3.3834 and occurs at $x=5$, and the absolute minimum value is 1.1851 and occurs at $x=12$.

5. Find all local extrema of each of the given functions on its domain. Use the Second Derivative Test when it applies.

(a) $f(x) = x^4 - 8x^3 - 32x^2 + 10$ Domain: $\mathbb{R}$

$f'(x) = 4x^3 - 24x^2 - 64x = 0$

$4x(x^2 - 6x - 16) = 0$

$4x(x-8)(x+2) = 0$

$x = 0 \quad x = 8 \quad x = -2$

(all are in domain)

Second Deriv. Test

$f''(x) = 12x^2 - 48x - 64$

$f''(0) = -64 < 0$ ∩

$\boxed{\bullet \text{ local maximum at } x = 0}$

$f''(8) = 12(8)^2 - 48(8) - 64 = 320 > 0$ ∪

$\boxed{\bullet \text{ local minimum at } x = 8}$

$f''(-2) = 12(-2)^2 - 48(-2) - 64 = 80 > 0$ ∪

$\boxed{\bullet \text{ local minimum at } x = -2}$

Coordinates

local max: $(0, 10)$

local mins: $(-2, -38)$ and $(8, -2038)$

(b) $g(x) = 0.5x^2 - 8x + 9\ln(x+2)$ given that $g'(x) = \dfrac{x^2 - 6x - 7}{x+2}$ and $g''(x) = \dfrac{x^2 + 4x - 5}{(x+2)^2}$

DOMAIN: $x > -2$ so $(-2, \infty)$

$g'(x) = \dfrac{(x-7)(x+1)}{x+2} = 0$

only if

$(x-7)(x+1) = 0$

$x = 7, \ x = -1$

(both are in the domain)

Second Deriv. Test

$g''(-1) = -8 < 0$ ∩

$\bullet\ \boxed{\text{local maximum at } x = -1}$

$g''(7) = \frac{8}{9} > 0$ ∪

$\bullet\ \boxed{\text{local minimum at } x = 7}$

Coordinates

local max: $(-1, 8.5)$

local min: $(7, -11.7250)$

(c) $h(x) = 2x^5 - 15x^4 - 90x^3 + 75$

Domain: $\mathbb{R}$

$h'(x) = 10x^4 - 60x^3 - 270x^2$

$= 10x^2(x^2 - 6x - 27) = 0$

$10x^2(x-9)(x+3) = 0$

$x = 0 \quad x = 9 \quad x = -3$

Sign Chart

$h'(x)$: $+++$ | -3 | $---$ | 0 | $---$ | 9 | $+++$

↑ no local extremum here (at $x = 0$)

Test #	$h'(x)$
-1	-200
1	-320
-4	2080
10	13000

$\boxed{\begin{array}{l}\text{Local maximum at } x = -3 \\ \text{Local minimum at } x = 9\end{array}}$

Second Deriv. Test

$h''(x) = 40x^3 - 180x^2 - 540x$

$h''(0) = 0$ Test fails!!! (must make sign chart)

$h''(9) = 9720 > 0$ ∪ local min

$h''(-3) = -1080 < 0$ ∩ local max

Coordinates

local max: $(-3, 804)$

local min: $(9, -45852)$

6. Each of the following functions has one absolute extremum on the provided interval. Find the location and value of the absolute extremum and classify it as an absolute maximum or absolute minimum.

(a) $f(x) = \frac{e^x}{x}$ on $(0, \infty)$

$$f'(x) = \frac{xe^x - e^x(1)}{x^2}$$

$$= \frac{e^x(x-1)}{x^2} = 0$$

only if

$e^x = 0$ or $x - 1 = 0$

no soln $x = 1$ in interval ✓

$$f''(x) = \frac{x^2\left[e^x(x-1) + e^x(1)\right] - e^x(x-1)(2x)}{x^4}$$

$$f''(x) = \frac{e^x \cdot x^2(x-1+1) - e^x(2x^2 - 2x)}{x^4}$$

$$f''(x) = \frac{e^x(x^3 - 2x^2 + 2x)}{x^4}$$

$$f''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$$

(Note: I changed the problem so that this would be given.)

$$f''(1) = \frac{e(1-2+2)}{1} = e > 0 \ \cup$$

Since $x=1$ was the only critical number in $(0,\infty)$ and $f''(1) > 0$, the absolute min. value occurs at $x=1$. The absolute minimum value is $f(1) = e$.

(b) $g(x) = 5x\ln x - 15x$ on $(0, \infty)$

$$g'(x) = 5\ln x + 5\cancel{x}\left(\tfrac{1}{\cancel{x}}\right) - 15$$

$$= 5\ln x + 5 - 15$$

$$g'(x) = 5\ln x - 10 = 0$$

$$5\ln x = 10$$

$$\ln x = 2$$

$x = e^2$ ← in domain ✓ (only critical #)

$$g''(x) = 5\left(\tfrac{1}{x}\right)$$

$$g''(e^2) = 5\left(\tfrac{1}{e^2}\right) = 0.6767 > 0 \ \cup$$

$$g(e^2) = 5(e^2)\ln(e^2) - 15(e^2)$$

$$= 10e^2 - 15e^2$$

$$= -5e^2 \approx -36.9453$$

The absolute minimum value of $g(x)$ on $(0,\infty)$ is -36.9453 and occurs at $x = e^2 \approx 7.3891$.

(c) $h(x) = 20 - 3x - \frac{12}{x^2}$ on $(0, \infty)$

$$h(x) = 20 - 3x - 12x^{-2}$$

$$h'(x) = -3 + 24x^{-3} \longrightarrow h''(x) = -72x^{-4}$$

$$= -3\tfrac{x^3}{x^3} + \frac{24}{x^3} = 0$$

$$\frac{-3x^3 + 24}{x^3} = 0$$

$$-3x^3 + 24 = 0$$

$$-3x^3 = -24$$

$$x^3 = 8$$

$$x = 2$$

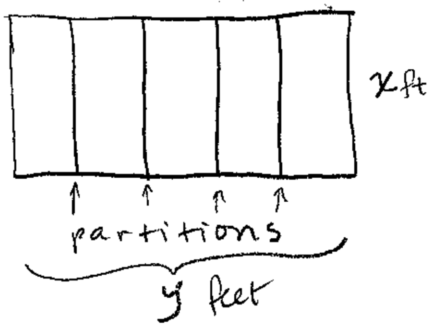
In interval ✓; only critical # in the interval

$$h''(2) = \frac{-72}{2^4} = -4.5 < 0 \ \cap$$

$$h(2) = 11$$

The absolute maximum value of $h(x)$ on $(0,\infty)$ is 11 and occurs at $x = 2$.

7. Bob would like to build a rectangular corral with four parallel partitions. If he plans to use 1,200 feet of fencing, what dimensions will maximize the total area of the corral?



$6x + 2y = 1200$

Maximize Area $A(x) = xy = x(600-3x)$

$2y = 1200 - 6x$

$y = 600 - 3x$

Domain: $x \geq 0$

$y = 600 - 3x \geq 0$

$-3x \geq -600$

$x \leq 200$

$[0, 200]$

$\boxed{\text{Maximize } A(x) = 600x - 3x^2 \text{ on } [0, 200]}$

$A'(x) = 600 - 6x = 0$

$-6x = -600$

$x = 100$

Closed Interval Method

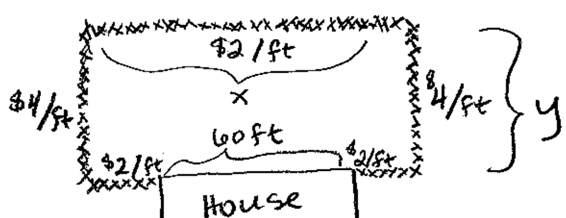
x	$A(x)$
0	0
100	30000 ← abs. max
200	0

$x = 100$ so $y = 600 - 3(100)$

$y = 300$

$\boxed{\text{The corral with maximum area is 100ft by 300ft.}}$

8. Now Bob wants to fence in a rectangular area of 4,232 ft^2 in his back yard. He would like for the fence to extend the same distance to the left and right of the back side of his house, which is 60 feet long. Material for the sides of the fence that are parallel to the back side of the house costs \$2 per foot, and material for the other two sides costs \$4 per foot. Find the dimensions of the fenced area that minimizes cost.



Minimize Cost $= C(x) = 2x + 2(x-60) + 4y + 4y$

$C(x) = 2x + 2x - 120 + 8y$

Area $= xy = 4232$

$y = \frac{4232}{x}$

$C(x) = 4x - 120 + 8\left(\frac{4232}{x}\right)$

$\boxed{\text{Minimize Cost } C(x) = 4x - 120 + \frac{33856}{x} \text{ on } [60, \infty)}$

Domain

Minimum x: $x = 60$

$x \geq 60$

$[60, \infty)$

not closed, so we must use 2nd deriv. test to confirm that $x = 92$ produces a minimum

$C(x) = 4x - 120 + 33856x^{-1}$

$C'(x) = 4 - 33856x^{-2}$

$= 4\frac{x^2}{x^2} - \frac{33856}{x^2} = 0$

$\frac{4x^2 - 33856}{x^2} = 0$

$4x^2 - 33856 = 0$

$4x^2 = 33856$

$x^2 = 8464$

$x = \pm 92$

$x = 92$ is in the domain

2nd Deriv. Test

$C''(x) = 67712x^{-3}$

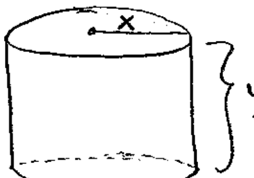
$C''(92) = \frac{2}{23} > 0$ ← abs. min

$x = 92$

$y = \frac{4232}{92} = 46$

$\boxed{\text{A 92ft by 46ft fenced area will minimize cost.}}$

9. Acme Water Supply wants to make a closed cylindrical water tank that has a volume of 225,000π cubic feet. The material for the top and bottom of the tank costs \$10 per square foot, and the material for the side costs \$6 per square foot. Find the dimensions (height and radius) of the water tank that minimizes cost.



Minimize Cost = cost of top + cost of bottom + cost of side

$= 10(\pi x^2) + 10(\pi x^2) + 6(2\pi x y)$

$C(x) = 20\pi x^2 + 12\pi x y = 20\pi x^2 + 12\pi x\left(\frac{225{,}000}{x^2}\right)$

Volume $= \pi x^2 y = 225{,}000\pi$ $\qquad C(x) = 20\pi x^2 + 2700000\pi x^{-1}$

$y = \frac{225{,}000}{x^2}$

Domain
$x > 0$
$(0, \infty)$

$\boxed{\text{Minimize } C(x) = 20\pi x^2 + 2700000\pi x^{-1} \text{ on } (0,\infty)}$

$x = 40.7163$
$y = \frac{225{,}000}{(40.7163)^2} = 135.7206$

$C'(x) = 40\pi x - 2700000\pi x^{-2} = 0$

$\frac{40\pi x^3 - 2700000\pi}{x^2} = 0$

$40\pi x^3 - 2700000\pi = 0$

$40\pi x^3 = 2700000\pi$

$x^3 = 67500$

$x = 67500^{1/3} = 40.7163$ In domain ✓

(Use 2nd Deriv. Test to confirm this produces a min)

The dimensions that minimize cost are height = 135.7206 ft, radius = 40.7163 ft.

10. The owner of a luxury motor yacht that sails among the 4,000 Greek islands charges \$600 per person per day if exactly 20 people sign up for the cruise. However, if more than 20 people sign up for the cruise, the price of each fare is reduced by \$4 for each additional passenger. Assuming at least 20 people sign up for the cruise, determine how many passengers will result in the maximum revenue for the owner of the yacht. What is the maximum revenue? What would be the fare per passenger in this case? (courtesy Jenn Whitfield)

Maximize Revenue = R(x) = selling price * # of units sold

$R(x) = (600 - 4x)(20 + x)$

$R(x) = 12000 + 520x - 4x^2$ where x is the # of additional passengers

Domain
price ≥ 0
$600 - 4x \geq 0$
$-4x \geq -600$
$x \leq 150$
$[0, 150]$

$\boxed{\text{Maximize } R(x) = 12000 + 520x - 4x^2 \text{ on } [0,150]}$

$R'(x) = 520 - 8x = 0$

$-8x = -520$

$x = 65$ additional passengers

Closed Interval

x	R(x)
0	12000
65	28900 ← max
150	0

Total # of passengers = 20 + 65 = 85

Max Revenue = \$28900

price = 600 − 4(65) = \$340

11. *Inventory Control* - Acme Jelly Company expects a uniform annual demand of ~~[illegible]~~ 49,000 jars of its jelly. Acme orders its jars from the Emca Bottling Company. For each order of jars placed, Emca charges a fee of \$150, and Acme's cost of storing empty jars for a year is \$0.30 per jar. How many jar orders should Acme place per year and how many jars should be in each shipment if the total cost from ordering and storage is to be minimized?

x = the number of jars in each order
y = the number of orders placed

$xy = 49000$
$y = \frac{49000}{x}$

Minimize Cost C(x) = storage cost + ordering cost

$C(x) = 0.30\left(\frac{x}{2}\right) + 150y$
$C(x) = 0.15x + 150\left(\frac{49000}{x}\right)$
$C(x) = 0.15x + 7350000x^{-1}$

Domain
min x: 1
max x: 49000
$[1, 49000]$

$$\boxed{\text{Minimize } C(x) = 0.15x + 7350000x^{-1} \text{ on } [1, 49000]}$$

$C'(x) = 0.15 - 7350000x^{-2} = 0$

$\frac{0.15x^2 - 7350000}{x^2} = 0$

$0.15x^2 - 7350000 = 0$

$x = \pm 7000$ (only x = 7000)

Closed Interval Method

x	C(x)
1	735,000.15
7000	1,155 ← min
49000	7,365

$x = 7000$
$y = \frac{49000}{7000} = 7$

$\boxed{\text{Acme should place 7 orders of 7000 jars ea. per year}}$

12. Apply the graphing strategy to sketch the graph of $f(x) = \frac{2x^2 + x - 15}{x^2 - 9}$.

- Domain: $x^2 - 9 = 0$
  $(x+3)(x-3) = 0$
  $x = \pm 3$
  All real numbers except $x = \pm 3$ or $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

- x-intercepts: $f(x) = 0$

$\frac{2x^2 + x - 15}{(x+3)(x-3)} = 0$

$\frac{(2x-5)\cancel{(x+3)}}{\cancel{(x+3)}(x-3)} = 0$ only if $2x - 5 = 0$

$2x = 5$
$x = \frac{5}{2}$
$\boxed{\left(\frac{5}{2}, 0\right)}$

x = −3 is the location of a hole

- y-int $f(0) = \frac{-15}{-9} = \frac{5}{3}$

$\boxed{\left(0, \frac{5}{3}\right)}$

hole: $(-3, 1.8333)$

$\lim_{x \to -3} \frac{2x-5}{x-3} = \frac{2(-3)-5}{-3-3} = 1.8333$

- vertical asymptotes: x = 3 only (hole at x = −3)

- horiz. asymp.: $\lim_{x \to \infty} \frac{2x^2 + x - 15}{x^2 - 9} = 2$ $\boxed{y = 2}$

continued...

$f(x) = \dfrac{2x-5}{x-3}$ $\quad x \neq \pm 3$

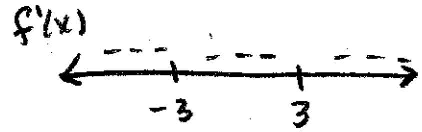
Note: $x=-3$ should be on the sign chart for $f'(x)$ because $f'(x)$ would not exist at $x=-3$ if you had found $f'(x)$ from the unsimplified version of $f(x)$.

$$f'(x) = \frac{(x-3)(2)-(2x-5)(1)}{(x-3)^2}$$

$$= \frac{2x-6-2x+5}{(x-3)^2}$$

$$f'(x) = \frac{-1}{(x-3)^2} < 0 \text{ for all } x$$

confirm:



Test #	$f'(x)$
-4	$-\frac{1}{49}$
0	$-\frac{1}{9}$
4	$-\frac{1}{49}$

$f(x)$ is decreasing on $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

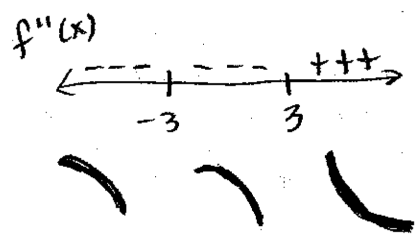
no local extrema

$$f'(x) = -(x-3)^{-2}$$

$$f''(x) = 2(x-3)^{-3}(1)$$

$$= \frac{2}{(x-3)^3} = 0$$

no soln



Test #	$f''(x)$
-4	$-\frac{2}{343}$
0	$-2/27$
4	2

$f(x)$ is concave down on $(-\infty,-3) \cup (-3,3)$

and concave up on $(3,\infty)$