Math 166 - Week in Review #8

Section 8.1 - Distributions of Random Variables

- A **random variable** is a rule that assigns a number to each oucome of a chance experiment.
- **Finite Discrete Random Variable** - A random variable is called finite discrete if it assumes only finitely many values. (You can write ALL possible values of the random variable in a list that stops.)
- **Infinite Discrete Random Variable** - A random variable is said to be infinite discrete if it takes on infinitely many values, which may be arranged in a sequence. (You can write all the possible values of the random variable in a list of numbers that has a pattern and goes on forever.)
- **Continuous Random Variable** - A random variable is called continuous if the values it may assume comprise an interval of real numbers. (For a continuous random variable, it is not possible to write an all-inclusive list of values.)
- **Histogram** - a graphical representation of a probability distribution of a random variable.

  Steps for Making a Histogram
  1. Locate the values of the random variable on the number line.
  2. Centered above each value of the random variable, make a rectangle with width 1 and height equal to the probability associated with that value of the random variable.

Section 8.2 - Expected Value

- Average, or Mean - The average, or mean, of the n numbers $x_1, x_2, \ldots, x_n$ is $\bar{x}$, where $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.
- Expected Value of a Random Variable X - Let X denote a random variable that assumes the values $x_1, x_2, \ldots, x_n$ with associated probabilities $p_1, p_2, \ldots, p_n$, respectively. Then the *expected value* of X, written $E(X)$, is given by the following formula: $E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$.
- If $P(E)$ is the probability of an event E occurring, then the odds in favor of E occurring are $\dfrac{P(E)}{P(E^c)}$, and the odds against E occurring are $\dfrac{P(E^c)}{P(E)}$. Whenever possible, odds are expressed as ratios of whole numbers. If the odds in favor of E are $\dfrac{a}{b}$, we say the odds in favor of E are a to b (or $a:b$). If the odds against E occurring are $\dfrac{b}{a}$, we say the odds against E are b to a (or $b:a$).
- If the odds in favor of an event E occurring are a to b, then the probability of E occurring is $P(E) = \dfrac{a}{a+b}$.
- **Median** - The median is the middle value in a set of data arranged in increasing or decreasing order (when there is an odd number of entries). If there is an even number of entries, the median is the average of the two middle numbers.
- **Mode** - The mode is the value that occurs most frequently in the set of data.

Section 8.3 - Variance and Standard Deviation

- **Variance of a Random Variable** X - The variance of a random variable X is one measure of dispersion (spread) of a probability distribution about its mean. The units of variance are the square of the units of the random variable.
- Standard Deviation of a Random Variable X - The standard deviation of a random variable X is another measure of dispersion (spread) of a probability distribution about its mean. The units of standard deviation are the same as the units of the random variable.
- Chebychev's Inequality - Let X be a random variable with expected value μ and standard deviation σ. Then the probability that a randomly chosen outcome of the experiment lies between $\mu - k\sigma$ and $\mu + k\sigma$ is at least $1 - \dfrac{1}{k^2}$; that is, $P(\text{outcome is within } k \text{ standard deviations of } \mu) \geq 1 - \frac{1}{k^2}$.

1. Classify each of the following random variables as either finite discrete, infinite discrete, or continuous, and give the values of the random variable.

(a) Cast a die until a 5 lands up. Let X denote the number of throws in one trial of the experiment.

$X = 1, 2, 3, 4, \ldots$ infinite discrete

(b) A farmer plants 10 watermelon seeds. Let X denote the number of the seeds that sprout.

$X = 0, 1, 2, 3, \ldots, 10$ finite discrete

(c) Let X denote the weight of my cat Mouse in pounds.

$X > 0$ continuous

(d) Let X denote the number of minutes that a person waits in line at a grocery store check-out lane during the 5 o'clock hour.

$0 \le X \le 60$ continuous

(e) A bag contains 3 red blocks, 2 green blocks, and 7 purple blocks. Blocks are drawn one at a time with replacement until a green block is drawn. Let X denote the number of blocks drawn in one trial of this experiment.

$X = 1, 2, 3, 4, \ldots$ infinite discrete

(f) Let X denote the number of queens in a 10-card hand that has been chosen from a standard deck of 52 playing cards.

$X = 0, 1, 2, 3, 4$ finite discrete

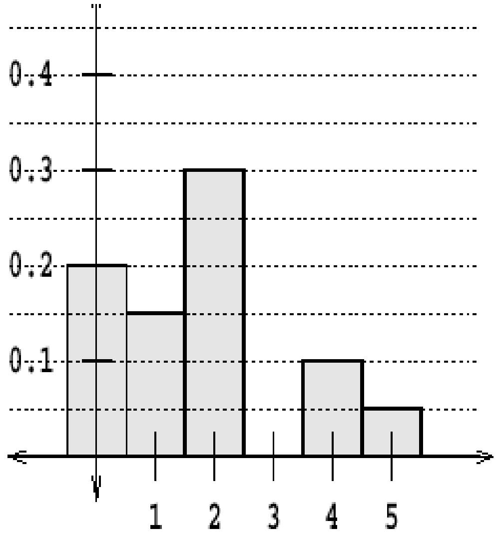
(g) A class consists of 2 freshmen, 5 sophomores, 9 juniors and 10 seniors. Students are selected one at a time at random from this class until a sophomore is found. No student gets selected more than once. Let X denote the number of students selected in one trial of this experiment.

$X = 1, 2, 3, \ldots, 22$

finite discrete

2. The following histogram gives the probability distribution for a random variable X which takes on the values 0, 1, 2, 3, 4, and 5.

(a) What are the odds that $X = 3$?



odds in favor: $\frac{P(E)}{P(E^c)}$

$P(X=3) = 1 - (.2 + .15 + .3 + .1 + .05)$
$= .2$

odds that $X=3$: $\frac{.2}{1-.2} = \frac{.2}{.8} = \frac{1}{4}$

$\boxed{1 \text{ to } 4}$

$1:4$

(b) Find $E(X)$.

$E(X) = 0(.2) + 1(.15) + 2(.3) + 3(.2) + 4(.1) + 5(.05)$

$\boxed{E(X) = 2}$

3. Let A and B be events of an experiment. The odds in favor of A occurring are 4 to 3, and the odds in favor of B occurring are 7 to 2. If the the odds in favor of A or B occurring are 19 to 2, are A and B independent events?

union

Test for Independence

Check: $P(A \cap B) \stackrel{?}{=} P(A)P(B)$?

Compute each separately:

$\boxed{P(A)} = \frac{4}{4+3} = \frac{4}{7}$

$\boxed{P(B)} = \frac{7}{7+2} = \frac{7}{9}$

$P(A \cup B) = \frac{19}{21}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{19}{21} = \frac{4}{7} + \frac{7}{9} - P(A \cap B)$

$\boxed{P(A \cap B)} = \frac{4}{9}$

Check: $\frac{4}{9} \stackrel{?}{=} \left(\frac{4}{7}\right)\left(\frac{7}{9}\right) = \frac{4}{9}$ ✓ (They are equal)

Answer: $\boxed{\text{A and B are independent.}}$

4. A box contains 2 pens, 3 pencils, and 5 highlighters. A sample of 6 items is drawn from the box without replacement. Let the random variable X denote the number of pencils in the sample.

(a) What type of random variable is X, and what are the possible values of X?

$X = 0, 1, 2, 3$ finite discrete

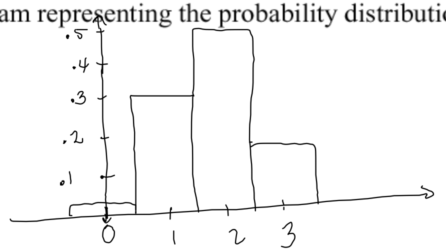
(b) Find the probability distribution of X.

Values of X	0	1	2	3
$P(X=x)$	$\frac{C(7,6)}{C(10,6)} \approx .0333 = \frac{1}{30}$	$\frac{C(3,1)C(7,5)}{C(10,6)} = .3 = \frac{3}{10}$	$\frac{C(3,2)C(7,4)}{C(10,6)} = .5 = \frac{1}{2}$	$\frac{C(3,3)C(7,3)}{C(10,6)} \approx .1667 = \frac{1}{6}$

$P(X=0) = \frac{n(X=0)}{n(S)}$

$P(X=1)$

(c) Draw a histogram representing the probability distribution of X.



(d) Find $P(X = 2)$.

$P(X=2) = .5$

(e) Find $P(X < 2)$.

$P(X<2) = P(X=0) + P(X=1)$

$= \frac{1}{30} + \frac{3}{10}$ ← from the table

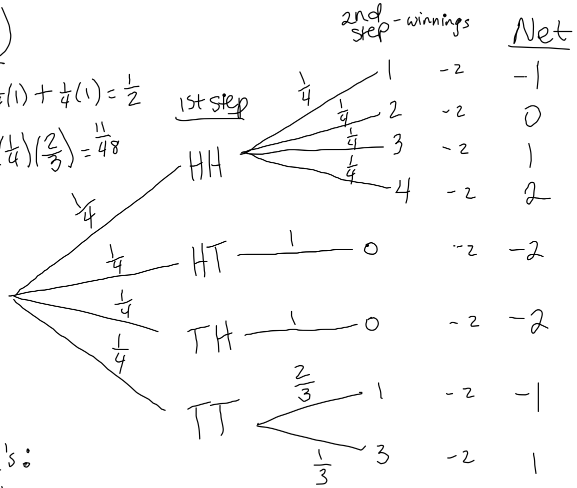
$= \boxed{\frac{1}{3}}$

5. A game consists of flipping two fair coins. If both coins land heads, then you draw 1 card from a bag that contains 4 cards, each marked with the number 1, 2, 3, or 4, and you win this amount in dollars. If both coins land tails, then you draw 1 block from a bucket that contains 3 blocks, two of which are labeled with a 1 and one of which is labeled with a 3, and you win this amount in dollars. Otherwise, you win nothing. It costs \$2 to play this game. Let X denote the net winnings of a person who plays this game once.

(a) Find the probability distribution of X. ← a table

X = net winnings (winnings − cost to play)

Values of X	P(X=x)
−2	$P(X=-2)=\frac{1}{4}(1)+\frac{1}{4}(1)=\frac{1}{2}$
−1	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)=\frac{11}{48}$
0	$\frac{1}{16}$
1	$\frac{7}{48}$
2	$\frac{1}{16}$



work for other probab's:

$P(X=0)=\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{1}{16}$

$P(X=1)=\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)=\frac{7}{48}$

$P(X=2)=\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{1}{16}$

(b) Find the expected net winnings of a person who plays this game once.

$E(X)=x_1p_1+x_2p_2+\cdots+x_np_n$

$=(-2)\left(\frac{1}{2}\right)+(-1)\left(\frac{11}{48}\right)+0\left(\frac{1}{16}\right)+1\left(\frac{7}{48}\right)+2\left(\frac{1}{16}\right)$

$=\boxed{-0.96}$

6. Acme Milk Distribution sells milk in 1-gallon jugs that have an actual volume that varies slightly from one jug to the next. The mean volume is 1.001 gallons with a standard deviation of 0.003 gallons. Use Chebychev's Theorem to estimate the probability that a randomly selected jug of Acme milk has a volume that is between 0.9944 and 1.0076 gallons.

X = volume of milk in a jug.

$P(.9944 \le X \le 1.0076) \ge 1 - \frac{1}{k^2}$ where

k = the number std. dev's away from the mean.

mean = $1.001 - .9944 = .0066$ difference

$1.0076 - 1.001$ (= mean) $= .0066$ difference

difference = σk (where σ = std. dev.)

$.0066 = .003k$

$2.2 = k$

$P(\text{within 2.2 std. dev's of the mean}) \ge 1 - \frac{1}{(2.2)^2} = .7934$

Answer: $\boxed{\text{at least } .7934}$

7. The odds that Bob will get a raise are 7 to 5. What is the probability that Bob will not get a raise?

1st Method

$$P(\text{not get a raise}) = \frac{5}{7+5} = \boxed{\frac{5}{12}}$$

2nd Method

$$P(\text{will get raise}) = \frac{7}{7+5} = \frac{7}{12}$$

$$P(\text{not get a raise}) = 1 - \frac{7}{12} = \boxed{\frac{5}{12}}$$

8. Bob plays the cello. Since he knows that he can be very clumsy (and also have bad luck), Bob decided that he should buy an insurance policy for his cello. What is the minimum premium that Bob can expect to pay for a $900 1-year policy if the insurance company estimates the probability that his cello will accidentally be damaged or stolen in 1 year to be 0.075?

minimum amt charged is the amount that would make the insurance company's expected gain = 0.

X = the gain of the insurance company

Want $E(X) = 0$.

1st make a probab. distrib

Let a = the minimum amt Bob can expect to pay.

	Value of X	$P(X=x)$	
(nothing happens to the cello)	a	.925	← 1-.075
something happened to the cello	$a-900$	.075	

$$E(X) = 0$$

$$a(.925) + (a-900)(.075) = 0$$

$$.925a + .075a - 67.5 = 0$$

$$a = 67.5$$

$\boxed{\text{Bob can expect to pay at least \$67.50.}}$

9. Suppose you roll two fair 6-sided dice and take the sum of the numbers landing up. You will win twice what you paid if the sum is 7 or 11. You win nothing if the sum is 2, 3, or 12. For any other sum, you win \$5. The game costs \$10 to play. Let X denote the net winnings of someone who plays once.

(a) What is the expected net winnings?

X = net winnings

Sum	Win	(net) Value of X	P(X=x)
7, 11	20 −10	10	$\frac{8}{36}$
2, 3, 12	0 −10	−10	$\frac{4}{36}$
any other possible sum	5 −10	−5	$\frac{24}{36}$

← $P(\text{sum is 7 or 11}) = \frac{2+6}{36} = \frac{8}{36}$

$P(\text{sum is 2, 3, or 12}) = \frac{1+1+2}{36} = \frac{4}{36}$

← $1 - \frac{8}{36} - \frac{4}{36}$

$$E(X) = 10\left(\frac{8}{36}\right) + (-10)\left(\frac{4}{36}\right) + (-5)\left(\frac{24}{36}\right)$$
$$= \boxed{-\$2.22}$$

(b) How much should be charged to make this game fair?

$E(X) = 0$

X = net winnings

Let a be the amt charged to make this game fair.

Sum	Win	(net) Value of X	P(X=x)
7 or 11	2a −a	a	$\frac{8}{36}$
2, 3, or 12	0 −a	−a	$\frac{4}{36}$
any other sum	5 −a	5−a	$\frac{24}{36}$

$$E(X) = 0$$

$$a\left(\frac{8}{36}\right) + (-a)\left(\frac{4}{36}\right) + (5-a)\left(\frac{24}{36}\right) = 0$$

$$\frac{8}{36}a - \frac{4}{36}a + \frac{10}{3} - \frac{24}{36}a = 0$$

$$-\frac{5}{9}a + \frac{10}{3} = 0$$

$$\frac{10}{3} = \frac{5}{9}a$$

$$\boxed{\$6} = a$$

You should charge \$6 to make the game fair.

10. A cashier at a convenience store kept a record of the number of items purchased by each customer on March 12. The data she collected are summarized in the following table:

sum of: 76

Number of customers	5	25	22	13	7	2	1	1
Number of items purchased	0	1	2	3	4	5	6	7

(a) What should the random variable X be defined as–number of customers or number of items purchased?

X= number of items purchased.

(b) Write the probability distribution of X.

Value of X	0	1	2	3	4	5	6	7
P(X=x)	$\frac{5}{76}$	$\frac{25}{76}$	$\frac{22}{76}$	$\frac{13}{76}$	$\frac{7}{76}$	$\frac{2}{76}$	$\frac{1}{76}$	$\frac{1}{76}$

(c) Find $P(2 \le X \le 6)$.

$P(2 \le X \le 6) = \frac{22}{76} + \frac{13}{76} + \frac{7}{76} + \frac{2}{76} + \frac{1}{76} = \frac{45}{76}$

(d) Find $P(X < 5)$.

$P(X<5) = 1 - \frac{2}{76} - \frac{1}{76} - \frac{1}{76} = \boxed{\frac{18}{19}}$

(e) How many items could a customer on that day be expected to buy?

$E(X) = \text{mean} = \boxed{2.0921}$

(f) Compute the mean, median, mode, standard deviation, and variance for the frequency chart. Be sure to label all answers.

Ramsey's class - This is a population.

mean = μ = 2.0921

median = 2

mode = 1

σ_X = std. dev = 1.3782

σ_X^2 = 1.8994 = variance

All other classes

mean = 2.0921

median = 2

mode = 1

σ_X = std. dev = 1.3782

σ_X^2 = 1.8994 = variance

(g) Use Chebychev's inequality to estimate the probability that the value of X is within 1.75 standard deviations of the mean.

$$P(\text{within } 1.75 \text{ std dev's of mean}) \geq 1 - \frac{1}{(1.75)^2} = \frac{33}{49}$$

$\approx .6735$

$\boxed{\text{at least } .6735}$

(h) Find the exact probability that the value of X is within 1.75 standard deviations of the mean for this data set.

mean $= 2.0921$

$\sigma_X = 1.3782$

mean $+ 1.75\sigma_X \rightarrow 2.0921 + 1.75(1.3782)$
$= 4.50395$

mean $- 1.75\sigma_X \rightarrow 2.0921 - 1.75(1.3782)$
$= -.31975$

$$P(\text{within } 1.75 \text{ std. dev's of mean}) = P(-.31975 \leq X \leq 4.50395)$$
$$= P(0 \leq X \leq 4)$$
$$= 1 - \frac{2}{76} - \frac{1}{76} - \frac{1}{76}$$
$$= \boxed{\frac{18}{19}} \approx .9474$$

11. The same cashier from the previous exercise kept a record of the number of items purchased by each customer on the next day. The data she collected are summarized in the following table:

Number of customers	2	13	20	33	18	4	3	0
Number of items purchased	0	1	2	3	4	5	6	7

(a) Let Y denote the number of items purchased by a customer on this day. Find $E(Y)$.

$E(Y) = 2.8172$
(from $\bar{x}$ in calc.)

(b) Find the variance and standard deviation of Y.

std. dev $= \sigma_Y = 1.2610$
Variance $\sigma_Y^2 = 1.5902$

(c) Refer to this and the previous exercise. Which random variable has more (spread) dispersion about its mean, i.e., which day's sales had more variability? Justify your answer.

$\sigma_X^2 = 1.8994$ and $\sigma_Y^2 = 1.5902$

X has more spread about its mean since the variance of X is larger than Y's.

12. Two marbles are drawn in succession without replacement from a bag containing 3 red, 6 blue, and 2 pink marbles. What are the odds that the second marble is blue given that the first marble was red?

Find odds

$$\text{odds in favor} = \frac{P(E)}{P(E^c)}$$

$$P(E) = \frac{6}{10}$$

where E is the event that the 2nd is blue given the 1st was red.

$$\text{odds in favor: } \frac{6/10}{1-6/10} = \frac{\frac{6}{10}}{4/10} = \frac{6}{4} = \frac{3}{2}$$

$$\text{odds} = \boxed{3 \text{ to } 2}$$