Math 166 - Exam 3 Review

NOTE: For reviews of the other sections on Exam 3, refer to the first page of WIR #7 and #8.

Section M.1 - Markov Processes

- **Markov Process (or Markov Chain)** - a special class of stochastic processes in which the probabilities associated with the outcomes at any stage of the experiment depend *only* on the outcomes of the preceding stage.
- The outcome at any stage of the experiment in a Markov process is called the **state** of the experiment.
- **Transition Matrix** - A transition matrix associated with a Markov chain with n states is an $n \times n$ matrix T with entries $a_{ij} = P(\text{moving to state } i | \text{ currently in state } j)$ such that
  1. $a_{ij} \geq 0$ for all i and j.
  2. The sum of the entries in each column of T is 1.
- Any matrix satisfying the two properties above is called a *stochastic* matrix.
- If T is the transition matrix associated with a Markov process, then the probability distribution of the system after n observations (or steps) is given by
$$X_n = T^n X_0$$
- **Powers of the Transition Matrix** - The entry of T^n in row i and column j gives the probability that the system moves to state i in n observations (i.e., n steps in the Markov chain) given that it was initially in state j.

Section M.2 - Regular Markov Processes

- The goal of this section is to investigate long-term trends of certain Markov processes.
- **Regular Markov Process** - A stochastic matrix T is a **regular Markov chain** if the sequence $T, T^2, T^3, \ldots$ approaches a steady state matrix in which all entries are *positive* (i.e., strictly greater than 0).
- It can be shown that a stochastic matrix T is regular if and only if *some* power of T has entries that are all positive.
- **Finding the Steady-State Distribution Vector** - Let T be a regular stochastic matrix. Then the steady-state distribution vector X may be found by solving the matrix equation $TX = X$ together with the condition that the sum of the elements of the vector X must equal 1.

Section M.3 - Absorbing Markov Processes

- **Absorbing Markov Process** - A Markov process is called absorbing if the following two conditions are satisfied.
  1. There is at least one absorbing state.
  2. It is possible to move from any nonabsorbing state to one of the absorbing states in a finite number of stages (i.e., steps in the Markov chain).
- The transition matrix for an absorbing Markov process is said to be an **absorbing stochastic matrix**. When studying the long-term behavior of an absorbing Markov process, we reorganize the transition matrix so that absorbing states are listed first, followed by nonabsorbing states.
- **Theorem 1** - In an absorbing Markov process, the long-term probability of going from any nonabsorbing state to some absorbing state is 1.

- **Theorem 2** (Tomastik and Epstein, pg. 350) - Let T be the transition matrix of an absorbing Markov process with a absorbing states and b nonabsorbing states. Reorder the states so that the first a of them are absorbing and the remaining b of them are nonabsorbing. Partition the matrix as follows:

$$\left[\begin{array}{c|c} I_{a\times a} & A_{a\times b} \\ \hline O_{b\times a} & B_{b\times b} \end{array}\right]$$

Then the following are true:

1. As n becomes large without bound, the matrix T^n, which is the transition matrix from the initial stage to the n^{th} stage, heads for the limiting matrix

$$L = \left[\begin{array}{c|c} I_{a\times a} & A(I-B)^{-1} \\ \hline O_{b\times a} & O_{b\times b} \end{array}\right]$$

where the identity matrix I in the expression $(I-B)^{-1}$ is the same dimension as B, that is, $b \times b$.

2. The entry in the i^{th} row and j^{th} column of the sub-matrix $A(I-B)^{-1}$ gives the probability that the system will end up in the i^{th} absorbing state when initially in the j^{th} nonabsorbing state.
3. All column sums of $A(I-B)^{-1}$ are 1, and thus everything is expected to be absorbed.

---

1. True/False

TRUE FALSE a) $I_nA = AI_n = A$ for all matrices A. True if A is nxn.

TRUE FALSE b) When using rref in the calculator, a row of all 0's at the bottom of the resulting matrix guarantees that the system of equations has infinitely many solutions.

TRUE FALSE c) To be able to compute the matrix product AB, the number of columns of A must equal the number of rows of B.

TRUE FALSE d) If B is a 2×2 matrix, then $B + I_2 = B$. $BI_2 = B$

TRUE FALSE e) If T is a 3×3 transition matrix, then t_{13} represents the probability of going to state 3 next if currently in state 1. t_{13} = probab. of next state being state 1 if current state is state 3.

TRUE FALSE f) If in solving a system of equations in the variables x and y using the Gauss-Jordan elimination method you obtain $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \end{array}\right]$, then the system of equations has only one solution.

TRUE FALSE g) If the parametric solution to a system of equations is $(3t+2, -t-3, t)$, then $(-7, 0, -3)$ is a particular solution. $t=-3$

TRUE FALSE h) If A is a 3×4 matrix and B is a 2×4 matrix, then BA^T is a 2×3 matrix.

Size of B Size of A^T

2x4 ✓= 4x3

2x3

2. For the next two word problems do the following:
   I) Define the variables that are used in setting up the system of equations.
   II) Set up the system of equations that represents this problem.
   III) Solve for the solution.
   IV) If the solution is parametric, then tell what restrictions should be placed on the parameter(s). Also give three specic solutions.

(a) Fred, Bob, and George are avid collectors of baseball cards. Among the three of them, they have 924 cards. Bob has three times as many cards as Fred, and George has 100 more cards than Fred and Bob do combined. How many cards do each of the friends have?

Let x = the number of baseball cards that Fred has.
Let y = the number of baseball cards that Bob has.
Let z = the number of baseball cards that George has.

$$x+y+z=924$$
$$y=3x$$
$$z=x+y+100$$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 924 \\ -3 & 1 & 0 & 0 \\ -1 & -1 & 1 & 100 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 103 \\ 0 & 1 & 0 & 309 \\ 0 & 0 & 1 & 512 \end{array}\right] \quad \begin{array}{l} x=103 \\ y=309 \\ z=512 \end{array}$$

Fred has 103 cards, Bob has 309 cards, and George has 512 cards.

(b) In a laboratory experiment, a researcher wants to provide a rabbit with exactly 1000 units of vitamin A, exactly 1600 units of vitamin C and exactly 2400 units of vitamin E. The rabbit is fed a mixture of three foods. Each gram of food 1 contains 2 units of vitamin A, 3 units of vitamin C, and 5 units of vitamin E. Each gram of food 2 contains 4 units of vitamin A, 7 units of vitamin C, and 9 units of vitamin E. Each gram of food 3 contains 6 units of vitamin A, 10 units of vitamin C, and 14 units of vitamin E. How many grams of each food should the rabbit be fed?

Let x = the number of grams of food 1.
Let y = the number of grams of food 2.
Let z = the number of grams of food 3.

	Food 1	Food 2	Food 3	Total
Vit A	2 units	4 units	6 units	1000 units
Vit C	3 units	7 units	10 units	1600 units
Vit E	5 units	9 units	14 units	2400 units

$$2x+4y+6z=1000$$
$$3x+7y+10z=1600$$
$$5x+9y+14z=2400$$

$$\begin{array}{c|c} x\ y\ z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 1000 \\ 3 & 7 & 10 & 1600 \\ 5 & 9 & 14 & 2400 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 300 \\ 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x+z=300$$
$$y+z=100$$

Let $z=t$, where t is any real number ≥ 0.

$$x=-z+300$$
$$x=-t+300$$
$$-t+300\geq 0$$
$$t\leq 300$$

$$y+z=100$$
$$y=-z+100$$
$$y=-t+100$$
$$-t+100\geq 0$$
$$t\leq 100$$

Parametric Solution

$(x,y,z)=(-t+300,-t+100,t)$ where $0\leq t\leq 100$.

Particular solutions:

- When $t=0$, $(x,y,z)=(300,100,0)$ so 300 grams of food 1, 100g of food 2, 0g of food 3
- When $t=100$, $(x,y,z)=(200,0,100)$ so 200g of food 1, 0g of food 2, 100g of food 3.
- When $t=1$, $(x,y,z)=(299,99,1)$ so 299g of food 1, 99g of food 2, 1g of food 3.

make a 1

3. Solve the system $\begin{aligned} 5x-3y &= 7 \\ 2x+6y &= -1 \end{aligned}$ using the Gauss-Jordan elimination method. (See WIR #7 for an example with three equations and three unknowns.)

$$\left[\begin{array}{cc|c} 5 & -3 & 7 \\ 2 & 6 & -1 \end{array}\right] \xrightarrow{R_1-2R_2\to R_1} \left[\begin{array}{cc|c} 1 & -15 & 9 \\ 2 & 6 & -1 \end{array}\right]$$

make a 0.

R_1-2R_2

$[5\ -3\ 7]-2[2\ 6\ -1]$

$=[5\ -3\ 7]+[-4\ -12\ 2]$

$=[1\ -15\ 9]$

$$\xrightarrow{R_2-2R_1\to R_2} \left[\begin{array}{cc|c} 1 & -15 & 9 \\ 0 & 36 & -19 \end{array}\right] \xrightarrow{\frac{1}{36}R_2\to R_2} \left[\begin{array}{cc|c} 1 & -15 & 9 \\ 0 & 1 & -\frac{19}{36} \end{array}\right]$$

make a 1

make a 0

Gauss-Jordan Elimination

Gauss Elimination

$x-15y=9$

$y=-\frac{19}{36}$

$x-15(-\frac{19}{36})=9$

$x=\frac{13}{12}$

$\boxed{(x,y)=(\frac{13}{12},-\frac{19}{36})}$

$$\xrightarrow{R_1+15R_2\to R_1} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & \frac{13}{12} \\ 0 & 1 & -\frac{19}{36} \end{array}\right] \end{array}$$

(Last step for Gauss-Jordan)

$x=\frac{13}{12}$

$y=-\frac{19}{36}$

$\boxed{(\frac{13}{12},-\frac{19}{36})}$

4. Solve for the variables x, y, z, and u. If this is not possible, explain why.

$2\times 2 \quad 2\times 2$

$$\begin{bmatrix} 2 & 0 \\ x & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ y+1 & z \end{bmatrix}+\begin{bmatrix} 7 & 0 \\ z & x \end{bmatrix}^T=-2\begin{bmatrix} 1-u & 5 \\ -3 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 8 \\ x-1(y+1) & 4x-z \end{bmatrix}+\begin{bmatrix} 7 & z \\ 0 & x \end{bmatrix}=\begin{bmatrix} -2+2u & -10 \\ 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 8+z \\ x-y-1 & 5x-z \end{bmatrix}=\begin{bmatrix} -2+2u & -10 \\ 6 & 3 \end{bmatrix}$$

$9=-2+2u$

$11=2u$

$\boxed{\frac{11}{2}=u}$

$8+z=-10$

$\boxed{z=-18}$

$5x-z=3$

$5x=z+3$

$5x=-18+3$

$5x=-15$

$\boxed{x=-3}$

$x-y-1=6$

$-y=7-x$

$y=x-7$

$y=-3-7$

$\boxed{y=-10}$

5. A small town has only two dry cleaners, Acme Dry Clean, and Emca Dry Cleaners. Acme's manager hopes to increase the firm's market share by conducting an extensive advertising campaign. After the campaign, a market research firm finds that 65% of Acme's customers will return to Acme Dry Clean with their next load of dry cleaning, and 25% of Emca's customers will switch to Acme for their next load of dry cleaning. Suppose that each customer brings one load of dry cleaning per week and that before the ad campaign, 35% of all customers used Acme Dry Clean and 65% of all customers used Emca Dry Cleaners.

(a) What is the transition matrix for this Markov process? If this is an absorbing Markov process, order the states so that all absorbing states are listed first.

A - Acme Dry Clean
E - Emca Dry Cleaners

Current

$$T = \underset{\text{Next}}{} \begin{array}{c} \\ A \\ E \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{bmatrix} .65 & .25 \\ .35 & .75 \end{bmatrix} \end{array}$$

(b) What will the percentages of the market share look like 4 weeks after the advertising campaign? Give your answer as percentages rounded to 2 decimal places.

$$X_0 = \begin{bmatrix} .35 \\ .65 \end{bmatrix} \qquad X_4 = T^4 X_0 = \begin{bmatrix} .41496 \\ .58504 \end{bmatrix}$$

$\boxed{\text{After 4 wks, 41.50\% will go to Acme Dry Clean and 58.50\% will go to Emca.}}$

(c) What is the probability that an individual who initially took his or her dry cleaning to Acme Dry Clean will then take his or her dry cleaning to Emca Dry Cleaners one week after the ad campaign begins?

current

$$\boxed{0.35} \qquad T^1 = \underset{\text{Next}}{} \begin{array}{c} \\ A \\ E \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{bmatrix} .65 & .25 \\ .35 & .75 \end{bmatrix} \end{array}$$

(d) What is the probability that someone who initially took his or her dry cleaning to Emca Dry Cleaners will take a load of dry cleaning to Emca six weeks after the ad campaign begins?

Current

$$T^6 = \underset{\text{Next (after 6)}}{} \begin{array}{c} \\ A \\ E \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{bmatrix} .4191 & .4150 \\ .5809 & .5850 \end{bmatrix} \end{array} \qquad \boxed{0.5850}$$

(e) Is this a regular Markov process or an absorbing Markov process? If regular, find the steady-state distribution vector for this Markov process. If absorbing, determine the long-term behavior of this Markov process. If neither, explain why.

Regular since all entries are probabilities, columns sum to 1, and T^1 has no 0's. Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the steady-state distrib. vector.

$$TX = X$$

$$\begin{bmatrix} .65 & .25 \\ .35 & .75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \boxed{X = \begin{bmatrix} 5/12 \\ 7/12 \end{bmatrix}}$$

$$\begin{aligned} 0.65x + 0.25y &= x \\ 0.35x + 0.75y &= y \end{aligned} \rightarrow \begin{aligned} -0.35x + 0.25y &= 0 \\ 0.35x - 0.25y &= 0 \\ x + y &= 1 \end{aligned} \quad \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} -.35 & .25 & 0 \\ .35 & -.25 & 0 \\ 1 & 1 & 1 \end{array}\right] \end{array} \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 5/12 \\ 0 & 1 & 7/12 \\ 0 & 0 & 0 \end{array}\right]$$

6. Use the given matrices to compute each of the following. If an operation is not possible, explain why.

$$A=\begin{bmatrix} -5 & 3 \\ 7 & 8 \end{bmatrix},\ B=\begin{bmatrix} 1 & 2 \\ 5 & -8 \\ 3 & 5 \end{bmatrix},\ C=\begin{bmatrix} 5 & 4 & 7 \\ 6 & -3 & 1 \end{bmatrix}$$

(a) $B+C$

$\underset{3\times 2}{B} + \underset{2\times 3}{C}$

B and C cannot be added because they are not the same size.

(b) BC

Size of B Size of C

$3\times 2 \overset{\checkmark}{=} 2\times 3$

3×3

$$\boxed{BC=\begin{bmatrix} 17 & -2 & 9 \\ -23 & 44 & 27 \\ 45 & -3 & 26 \end{bmatrix}}$$

(c) $AB^T - 5C$

Size of A Size of B^T − $5C$

$2\times 2 \overset{\checkmark}{=} 2\times 3$ $\quad$ 2×3

2×3 same ✓

$$\boxed{AB^T-5C=\begin{bmatrix} -24 & -69 & -35 \\ -7 & -14 & 56 \end{bmatrix}}$$

(d) CA

Size of C size of A

$2\times 3 \neq 2\times 2$

Not possible since the number of columns of C does not equal the number of rows of A.

7. (Inspired by Application 2, pg. 40 of *Linear Algebra with Applications*, 5th ed., by Steven J. Leon)

A company manufactures three products: A, B, and C. Its production expenses per item are summarized in the first table below. The second table gives the number of each type of product produced in each quarter of 2007.

Production Costs Per Item (in dollars)

	Product		
Expenses	A	B	C
Materials	20	26	17
Assembly	12	15	10
Packaging	2	1	2

Quantity Produced Each Quarter

	Quarter			
Product	1	2	3	4
A	350	375	370	365
B	320	290	275	250
C	485	410	415	390

Write two matrices C and Q summarizing the above information and show how matrix multiplication can be used to produce a matrix that gives the total costs for each type of expense for each quarter.

$$C=\begin{array}{r} \\ \text{mater.} \\ \text{Assem.} \\ \text{Pack.} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 20 & 26 & 17 \\ 12 & 15 & 10 \\ 2 & 1 & 2 \end{bmatrix} \end{array} \qquad Q=\begin{array}{r} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 350 & 375 & 370 & 365 \\ 320 & 290 & 275 & 250 \\ 485 & 410 & 415 & 390 \end{bmatrix} \end{array}$$

Example: Total material cost for Quarter 1: $20*350+26*320+17*485$

$$\boxed{CQ}=\begin{array}{r} \\ \text{mater.} \\ \text{Assem.} \\ \text{Pack.} \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 23565 & 22010 & 21605 & 20430 \\ 13850 & 12950 & 12715 & 12030 \\ 1990 & 1860 & 1845 & 1760 \end{bmatrix} \end{array}$$

8. (adaptation of #11, pg. 427 of *Finite Mathematics and its Applications*, 9th ed, by Goldstein, Schneider, and Siegel)

A group of physical fitness devotees works out in the gym every day. The workouts vary from strenuous to moderate to light. When their exercise routine was recorded, the following observation was made: Of the people who work out strenuously on a particular day, 40% will work out strenuously the next day, 45% will work out moderately the next day, and the rest will do a light workout the next day. Of the people who work out moderately on a particular day, 50%, 30% and 20% will work out strenuously, moderately, and lightly (respectively) the next day. Of the people working out lightly on a particular day, 45%, 30% and 25% will work out strenuously, moderately, and lightly (respectively) the next day.

(a) Write the transition matrix for this Markov process. If this is an absorbing Markov process, order the states so that all absorbing states are listed first.

S-strenuous workout
M-moderate workout
L-light workout

current

$$T = \begin{array}{c|ccc} & S & M & L \\ \hline S & .4 & .5 & .45 \\ M & .45 & .3 & .30 \\ L & .15 & .2 & .25 \end{array}$$

Next state

(b) What is the probability that a person who did a $\underbrace{\text{strenuous}}_{\text{column 1}}$ workout on $\underbrace{\text{Tuesday}}_{\text{time 0}}$ will do a $\underbrace{\text{light}}_{\text{row 3}}$ workout on $\underbrace{\text{Friday}}_{\text{3 days later}}$ (same week)?

Look at the row 3, column 1 entry of T^3: $\boxed{0.18675}$

or $X_0 = \begin{array}{c} S \\ M \\ L \end{array}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $X_3 = T^3 X_0 = \begin{array}{c} S \\ M \\ L \end{array}\begin{bmatrix} .44536 \\ .36788 \\ .18675 \end{bmatrix}$

(c) Suppose that on a particular Monday 80% do a strenuous workout, 10% do a moderate workout, and 10% do a light workout. What will the percentages for each type of workout look like on Wednesday of the following week?

$X_0 = \begin{array}{c} S \\ M \\ L \end{array}\begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix}$ $\quad X_9 = T^9 X_0 = \begin{bmatrix} .4460 \\ .3669 \\ .1871 \end{bmatrix}$

On Wednesday of next week, 44.60% will do a strenuous workout, 36.69% will do a moderate workout, and 18.71% will do a light workout.

(d) Is this a regular Markov process or an absorbing Markov process? If regular, find the steady-state distribution vector for this Markov process. If absorbing, determine the long-term behavior of this Markov process. If neither, explain why.

Regular since all entries of T are positive. Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the steady-state distribution vector.

$TX = X$

$$\begin{bmatrix} .4 & .5 & .45 \\ .45 & .3 & .3 \\ .15 & .2 & .25 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{aligned} .4x + .5y + .45z &= x \\ .45x + .3y + .3z &= y \\ .15x + .2y + .25z &= z \end{aligned} \rightarrow \begin{aligned} -.6x + .5y + .45z &= 0 \\ .45x - .7y + .3z &= 0 \\ .15x + .2y - .75z &= 0 \\ x + y + z &= 1 \end{aligned}$$

$$\begin{array}{ccc|c} x & y & z & \\ -.6 & .5 & .45 & 0 \\ .45 & -.7 & .3 & 0 \\ .15 & .2 & -.75 & 0 \\ 1 & 1 & 1 & 1 \end{array} \xrightarrow{\text{rref}} \begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 0 & 62/139 \\ 0 & 1 & 0 & 51/139 \\ 0 & 0 & 1 & 26/139 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\boxed{X = \begin{bmatrix} 62/139 \\ 51/139 \\ 26/139 \end{bmatrix}}$$

9. Solve the following systems of equations. If there are infinitely many solutions, state so and give the parametric solution. If there is no solution, state so.

(a) $$\begin{aligned} -3x &= 4y+z \\ y-7 &= -x+5z \\ 2x+z &= 14-y \end{aligned}$$

$$\begin{aligned} -3x-4y-z&=0 \\ x+y-5z&=7 \\ 2x+y+z&=14 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & \\ \end{array} \left[\begin{array}{ccc|c} -3 & -4 & -1 & 0 \\ 1 & 1 & -5 & 7 \\ 2 & 1 & 1 & 14 \end{array}\right] \xrightarrow{rref} \begin{array}{ccc} x & y & z \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 35/3 \\ 0 & 1 & 0 & -77/9 \\ 0 & 0 & 1 & -7/9 \end{array}\right]$$

$$\boxed{(x,y,z)=\left(\frac{35}{3}, -\frac{77}{9}, -\frac{7}{9}\right)}$$

(one solution)

(b) $$\begin{aligned} 9x+2y-6z &= 2 \\ 3x-y-2z &= 7 \\ -3x-4y+2z &= 15 \\ 6x-2y-4z &= 17 \end{aligned}$$

$$\begin{array}{ccc} x & y & z \end{array} \left[\begin{array}{ccc|c} 9 & 2 & -6 & 2 \\ 3 & -1 & -2 & 7 \\ -3 & -4 & 2 & 15 \\ 6 & -2 & -4 & 17 \end{array}\right] \xrightarrow{rref} \begin{array}{ccc} x & y & z \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0=1$ False!

$0=0$ true

$\boxed{\text{No Solution}}$

(c) $$\begin{aligned} 2x-y &= -7 \\ 6x-3y+4z &= -20 \end{aligned}$$

$$\begin{array}{ccc} x & y & z \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 0 & -7 \\ 6 & -3 & 4 & -20 \end{array}\right] \xrightarrow{rref} \begin{array}{ccc} x & y & z \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & -7/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

$x-\frac{1}{2}y=-\frac{7}{2}$

$z=\frac{1}{4}$

Let $y=t$ where t is any real number.

$x=\frac{1}{2}y-\frac{7}{2}$

$x=\frac{1}{2}t-\frac{7}{2}$

$z=\frac{1}{4}$

$$\boxed{(x,y,z)=\left(\frac{1}{2}t-\frac{7}{2}, t, \frac{1}{4}\right)}$$

Infinitely many solutions

10. Determine whether each of the following is the transition matrix for a regular Markov process, an absorbing Markov process, or neither. If the transition matrix is for an absorbing Markov process, reorder the states so that all abosorbing states are first, and then find the limiting matrix.

(a) $\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0.7 & 0.3 \\ 1 & 0.2 & 0.1 \\ 0 & 0.1 & 0.6 \end{bmatrix} \end{array} = T$

A is not an absorbing state.

Look at T^2: $T^2 = \begin{bmatrix} .7 & .17 & .25 \\ .2 & .75 & .38 \\ .1 & .08 & .37 \end{bmatrix}$

Since all entries of T^2 are positive, T is regular.

(b) $\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = T$

None of these states are absorbing

Look at powers of T: $T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $T^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$T^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = T$

Since no power of T has all positive entries, T is not regular. Since T has no absorbing states, it is also not absorbing.

(c) $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.3 & 0 & 0 & 0.6 \\ 0.1 & 1 & 0 & 0 \\ 0.1 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0.4 \end{bmatrix} \end{array} = T$

Once in C, always in C.

Once in B, always in B

T is not regular since no power of T has all positive entries. However, T is absorbing since it has two absorbing states (B and C), and it is possible to get from any nonabsorbing state (A or D) to an absorbing state in a finite number of steps.

Reordered

current

$T = \begin{array}{c} \\ B \\ C \\ A \\ D \end{array}\begin{array}{c} \begin{array}{cccc} B & C & A & D \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0.1 & 0 \\ \hline 0 & 0 & 0.3 & 0.6 \\ 0 & 0 & 0.5 & 0.4 \end{array}\right] \end{array}$

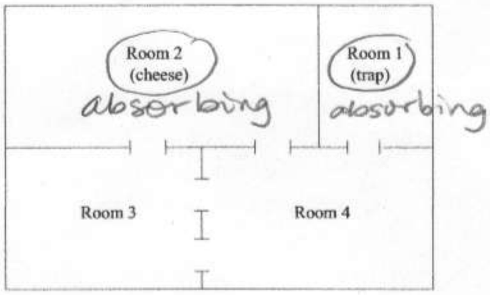
Next State

Matrix A

Matrix B

$L = \left[\begin{array}{c|c} I_{2x2} & A(I-B)^{-1} \\ \hline O_{2x2} & O_{2x2} \end{array}\right] = \begin{array}{c} \\ B \\ C \\ A \\ D \end{array}\begin{array}{c} \begin{array}{cccc} B & C & A & D \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 & 0.5 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

11. (Adapted from #12, pg. 450 of *Finite Mathematics and its Applications*, 9th ed, by Goldstein, Schneider, and Siegel)
The figure below gives the layout of a house with four rooms connected by doors. Room 1 contains a (live) mousetrap, and Room 2 contains cheese. A mouse, after being placed in either Room 3 or 4, will search for cheese; if unsuccessful after two minutes, it will exit to another room by selected one of the doors at random. A mouse entering Room 1 will be trapped and therefore no longer move to other rooms. Also, a mouse entering Room 2 will remain in that room.



(a) Write the transition matrix for this Markov process. If this is an absorbing Markov process, order the states so that all absorbing states are listed first. States listed by room number.

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{3} & \frac{1}{4} \\ \hline 0 & 0 & 0 & \frac{2}{4} \\ 0 & 0 & \frac{2}{3} & 0 \end{array}\right] \end{array}$$

(b) If a mouse $\underbrace{\text{begins in Room 3}}_{\text{column 3}}$, what is the probability that it will $\underbrace{\text{find the cheese}}_{\text{row 2}}$ $\underbrace{\text{after 8 minutes}}_{\text{8 min = 4 steps in Markov chain.}}$?

Look at row 2, column 3 of T^4: $\boxed{\frac{2}{3}}$

or $X_0 = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $X_4 = T^4 X_0 = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 2/9 \\ 2/3 \\ 1/9 \\ 0 \end{bmatrix}$

(c) Determine the long-term behavior of this Markov process by finding the limiting matrix and explaining the meaning of its entries.

$$L = \left[\begin{array}{c|c} I_{2\times2} & A(I_2 - B)^{-1} \\ \hline O_{2\times2} & O_{2\times2} \end{array}\right] \quad \text{where } A = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \frac{2}{4} \\ \frac{2}{3} & 0 \end{bmatrix}.$$

$$L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1/4 & 3/8 \\ 0 & 1 & 3/4 & 5/8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

In the long run, if the mouse starts in room 3, then the probab. is $\frac{1}{4}$ that it will eventually be caught in the trap and $\frac{3}{4}$ that it will eventually find the cheese. If the mouse starts in room 4, the probability is $3/8$ that it will eventually be caught in the trap and $5/8$ that it will eventually find the cheese.