LECTURE # 4:
INF-SUP CONDITION AND
BANACH-NEČAS-BABUŠKA THEOREM

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1. Abstract Weak Problems and Their FE Approximations

1.1. **Abstract Problem.** Now we shall consider the following abstract problem. Consider the Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ with norms generated by the corresponding inner (scalar) products: $(U,V)_{\mathcal{V}}$ for $U,V \in \mathcal{V}$ and $(W,Z)_{\mathcal{W}}$ for $W,Z \in \mathcal{W}$. Next, we introduce the bilinear form $\mathcal{A}(V,W): \ \mathcal{V}\times\mathcal{W} \to \mathbf{R}$ and the linear form $\mathcal{L}(W): \ \mathcal{W} \to \mathbf{R}$. Now consider the following abstract problem:

$$\text{find } U \in \mathcal{V} \text{ such that } \quad \mathcal{A}(U,W) = \mathcal{L}(W) \quad \forall W \in \mathcal{W}. \tag{1}$$

The aim of this lecture is to introduce an abstract framework to study this class of problems and discuss the existence and uniqueness of the solution and its continuous dependence on the data.

In order to illustrate this concept, we first give a number of examples.

Ex # 1: Take $U = u$, $\mathcal{V} = \mathcal{W} = H_0^1(\Omega)$,

$$\mathcal{A}(u,w) := \int_\Omega \nabla u \cdot \nabla w dx \text{ and } \mathcal{L}(w) = \int_\Omega f w dx,$$

with $f(x)$ given function in $L^2(\Omega)$. As we know from the previous lectures this is the weak form of the problem $-\Delta u = f$ for $x \in \Omega$ and $u(x) = 0$ for $x \in \partial\Omega$.

Ex # 2: (indefinite elliptic problem). Take $U = u$, $\mathcal{V} = \mathcal{W} = H_0^1(\Omega)$,

$$\mathcal{A}(u,w) := \int_\Omega (\nabla u \cdot \nabla w - \omega^2 u w) dx \text{ and } \mathcal{L}(w) = \int_\Omega f w dx$$

with $f(x)$ given function in $L^2(\Omega)$. This is the weak form of the Helmholtz equation $-\Delta u - \omega^2 u = f$ for $x \in \Omega$ with Dirichlet boundary conditions $u(x) = 0$ for $x \in \partial\Omega$.

Ex # 3: (indefinite mixed problem corresponding to second order differential equations). Take $U = (\mathbf{q}, u)$, $W = (\mathbf{r}, w)$, $\mathcal{V} = \mathcal{W} = H(div;\Omega) \times L^2(\Omega)$ and

$$\mathcal{A}(U,W) := \int_\Omega \mathbf{q}\cdot\mathbf{r} dx - \int_\Omega u \nabla\cdot\mathbf{r} dx - \int_\Omega \nabla \mathbf{q} w dx,$$

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and

$$\mathcal{L}(W) := l(w) = \int_\Omega f w dx.$$

This is the weak form of the mixed system corresponding to example #4 considered in Lecture # 0 with $K(x) = I$, $c(x) = 0$.

Ex # 4: Stokes system (example # 13 of Lecture # 0) in a weak form. In this case we set $U = (\mathbf{u}, p)$, $\mathcal{V} = \mathcal{W} = H_0^1(\Omega) \times L_0^2(\Omega)$ (with $L_0^2(\Omega)$ functions in $L^2(\Omega)$ with zero mean value on Ω) and seek a pair $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ that satisfies the integral identity

$$\mathcal{A}(U, W) := \mathcal{A}(\mathbf{u}, p; \mathbf{v}, q) := (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{u}, q) = (f, \mathbf{v})$$

for all $(\mathbf{v}, q) \in H_0^1(\Omega) \times L_0^2(\Omega)$.

Now we introduce the main tool for our analysis. This is not the most general form, but it is enough for our purposes. The most general theorem is valid for general Banach spaces.

Theorem 1. *(inf-sup) (Banach-Nečas-Babuška, Ladyzhenskaya-Babuška-Brezzi, Babuška-Brezzi) Let $\mathcal{V}$ and $\mathcal{W}$ be Hilbert spaces. The problem* (1) *is well-posed if and only if:*

$$\text{(BNB1)} \qquad \forall U \in \mathcal{V} \qquad \Big(\mathcal{A}(U, W) = 0 \ \forall W \in \mathcal{W}\Big) \ \textit{implies}\ U = 0;$$

$$\text{(BNB2)} \qquad \exists \alpha_0 > 0 : \qquad \sup_{W \in \mathcal{W}} \frac{\mathcal{A}(U, W)}{\|W\|_{\mathcal{W}}} \geq \alpha_0 \|U\|_{\mathcal{V}} \quad \forall U \in \mathcal{V}.$$

Moreover, the following a priori estimate holds

$$\|U\|_{\mathcal{V}} \leq \frac{1}{\alpha_0} \sup_{W \in \mathcal{W}} \frac{\mathcal{L}(w)}{\|W\|_{\mathcal{W}}} := \frac{1}{\alpha_0} \|\mathcal{L}\|_{\mathcal{W}'}.$$

Here $\|\mathcal{L}\|_{\mathcal{W}'}$ is the norm of $\mathcal{L}$ in the dual $\mathcal{W}'$ of $\mathcal{W}$.

We are not going to prove this theorem but we shall illustrate how to use it for various boundary value problems for PDEs.

First, we note that if the bilinear form is defined on $\mathcal{V} \times \mathcal{V}$ (in this case $\mathcal{V} = \mathcal{W}$) and is coercive in $\mathcal{V}$ then it satisfies the conditions of BNB Theorem. Indeed, coercivity in $\mathcal{V}$ means that

$$\mathcal{A}(U, U) \geq \alpha_0 \|U\|_{\mathcal{V}}^2, \quad \alpha_0 = const > 0.$$

Then (BNB1) follows immediately: if $\mathcal{A}(U, W) = 0 \ \forall W \in \mathcal{W}$ holds then choosing $W = U$ and using the coercivity we get $0 = \mathcal{A}(U, U) \geq \alpha_0 \|U\|_{\mathcal{V}} \geq 0$ implies that $\|U\|_{\mathcal{V}} = 0$ and therefore $U = 0$.

The inf-sup condition (BNB2) follows easily as well. By choosing $W = U$ we get

$$\sup_{W \in \mathcal{V}} \frac{\mathcal{A}(U, W)}{\|W\|_{\mathcal{V}}} \geq \frac{\mathcal{A}(U, U)}{\|U\|_{\mathcal{V}}} \geq \frac{\alpha_0 \|U\|_{\mathcal{V}}^2}{\|U\|_{\mathcal{V}}} = \alpha_0 \|U\|_{\mathcal{V}}.$$

1.2. **Abstract Galerkin Method.** Now we shall consider the Galerkin method for the problem (1). Let $\mathcal{V}_h \subset \mathcal{V}$ and $\mathcal{W}_h \subset \mathcal{W}$ be finite dimensional. Later we shall give particular examples of such spaces for the case of second order elliptic problem. Then the abstract Ritz-Galerkin method is:

$$(V_h) \qquad \boxed{\text{find } U_h \in \mathcal{V}_h \text{ such that } \quad \mathcal{A}(U_h, W) = \mathcal{L}(W) \quad \forall W \in \mathcal{W}_h.}$$

Now we introduce the discrete *inf-sup* condition:

$$(\text{BNB2}_h) \qquad \exists \alpha_0^* > 0 : \qquad \sup_{W \in \mathcal{W}_h} \frac{\mathcal{A}(U_h, W)}{\|W\|_{\mathcal{W}}} \geq \alpha_0^* \|U_h\|_{\mathcal{V}} \quad \forall U_h \in \mathcal{V}_h.$$

Remark 1. *Note that since $w \in \mathcal{W}_h$ the supremum is taken in a much smaller set and the discrete inf-sup condition* (BNB2_h) *does not follow from the infinite dimensional case* (BNB2)*. For each particular choice of the spaces $\mathcal{V}_h$ and $\mathcal{W}_h$ we have to verify the condition* (BNB2_h)*.*

Now we can prove the Cea's Lemma for discrete problems with indefinite bilinear forms.

Lemma 1. *(Cea's lemma) Assume the discrete inf-sup condition* (BNB2_h) *is satisfied. Then the abstract Galerkin method* (V_h) *has unique solution $U_h \in \mathcal{V}_h$.*

If in addition the bilinear form $\mathcal{A}(U, W)$ is continuous with respect to both $U \in \mathcal{V}$ and $W \in \mathcal{W}$, i.e. there is a constant c_0 such that

$$\mathcal{A}(U, W) \leq c_0 \|U\|_{\mathcal{V}} \, \|W\|_{\mathcal{W}}, \tag{2}$$

then this solution satisfies the estimate

$$\|U - U_h\|_{\mathcal{V}} \leq \left(1 + \frac{c_0}{\alpha_0^*}\right) \inf_{V \in \mathcal{V}_h} \|U - V\|_{\mathcal{V}}. \tag{3}$$

Proof. We first note that the finite dimensional problem (V_h) is a square system of linear algebraic equations for the unknown degrees of freedom. Then a discrete variant of the condition (BNB1) is equivalent to non-singularity of the corresponding matrix. The discrete variant of the *inf-sup* condition (BNB2) is equivalent to the non-singularity of the transposed matrix. But as we know, for square matrices non-singularity of the matrix is equivalent to the non-singularity of its transposed. Therefore in the finite dimensional case we need only the *inf-sup* condition. Note that *inf-sup* conditions ensures more than the non-singularity of the matrix, it guarantees that the lowest eigenvalue is bounded away from 0 uniformly in h.

Since the matrix of the corresponding system is non-singular, the problem (V_h) has unique solution U_h. Also $\mathcal{W}_h \subset \mathcal{W}$ and in (1) we can take $W \in \mathcal{W}_h$ to get

$$\mathcal{A}(U, W) = \mathcal{L}(W) \quad \text{and} \quad \mathcal{A}(U_h, W) = \mathcal{L}(W) \quad \forall W \in \mathcal{W}_h,$$

which leads to the well-known Galerkin orthogonality

$$\mathcal{A}(U - U_h, W) = 0 \qquad \forall W \in \mathcal{W}_h.$$

Now let V be an arbitrary function in $\mathcal{V}_h$. Then by the discrete *inf-sup* condition (BNB2$_h$) and by Galerkin orthogonality we have

$$\alpha_0^*\|U_h - V\| \le \sup_{W\in\mathcal{W}_h} \frac{\mathcal{A}(U_h - V, W)}{\|W\|_{\mathcal{W}}} = \sup_{W\in\mathcal{W}_h} \frac{\mathcal{A}(U - V, W)}{\|W\|_{\mathcal{W}}}.$$

The continuity (2) ensures that $\mathcal{A}(U - V, W) \le c_0\|U - V\|_{\mathcal{V}}\|W\|_{\mathcal{W}}$ and therefore

$$\alpha_0^*\|U_h - V\| \le c_0\|U - V\|_{\mathcal{V}} \quad \text{for any} \quad V \in \mathcal{V}_h.$$

Now by triangle inequality and the above estimate we get for any $V \in \mathcal{V}$:

$$\|U - U_h\|_{\mathcal{V}} \le \|U - V\|_{\mathcal{V}} + \|V - U_h\|_{\mathcal{V}} \le \Big(1 + \frac{c_0}{\alpha_0^*}\Big)\|U - V\|_{\mathcal{V}}.$$

Taking infimum in $V \in \mathcal{V}_h$ we get the desired result. □

2. Application to Indefinite Elliptic Problems

Now let us illustrate the use of this theorem on Example 2 by choosing $\Omega = (0,1)$ and taking $\omega = 5$. Then the problem is: find $u \in H_0^1(0,1)$ such that

$$\mathcal{A}(u,w) := \int_0^1 (u'w' - 25uw)dx = \int_0^1 fw\,dx \quad \forall w \in H_0^1(0,1).$$

Obviously, taking $u(x) = \sin(\pi x)$ (which is obviously in $H_0^1(0,1)$ we get $\mathcal{A}(u,u) = \int_0^1(\pi^2\cos^2(\pi x) - 25\sin^2(\pi x))dx < 0$. Therefore, the form is not coercive in $H_0^1(0,1)$. However, this problem is well posed, a result that will follow from BNB Theorem.

The condition (BNB1) is obviously satisfied since $\mathcal{A}(u,w) = 0$ for all $w \in H_0^1(0,1)$ is equivalent to $-u''(x) = 25u$ and $u(0) = u(1) = 0$. We know that the problem $-u''(x) = \lambda^2 u$ and $u(0) = u(1) = 0$ has infinitely many solution $u_n = \sin(\lambda_n x), \lambda_n = \pi n$, $n = 1, 2, \ldots$ called eigenpairs, and these are all nontrivial solutions of the problem. But $5 \ne \lambda_n$ for any n and therefore the above problem must have the solution $u(x) \equiv 0$, this means that the condition (BNB1) is satisfied.

More difficult is to show the *inf-sup* condition (BNB2). This we shall do in the more general case of indefinite problems. This will follow from the more general result we shall state below.

Theorem 2. *Assume that the bilinear form $\mathcal{A}(u,v)$ defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ satisfies:*

(a) *Görding inequality, i.e. for some constants $\alpha > 0$ and $\beta > 0$*

$$\mathcal{A}(u,u) \ge \alpha\|u\|_{H^1}^2 - \beta\|u\|_{L^2}^2 \quad \forall u \in H_0^1; \tag{4}$$

(b) *if $u \in H_0^1(\Omega)$ satisfy $\mathcal{A}(u,v) = 0$ $\forall v \in H_0^1(\Omega)$ then $u \equiv 0$.*

Then there is a constant $\alpha_0 > 0$ such that

$$\sup_{v\in H_0^1} \frac{\mathcal{A}(u,v)}{\|v\|_{H_0^1}} \ge \alpha_0\|u\|_{H_0^1}. \tag{5}$$

Before proving this theorem let us apply it to the Helmholtz problem of Example # 2 with $\mathcal{A}(u,v) = (\nabla u, \nabla v) - \omega^2(u,v)$. Then Görding inequality follows immediately

$$\mathcal{A}(u,u) \geq \int_\Omega (|\nabla u|^2 + u^2)dx - (\omega^2+1)\int_\Omega u^2 dx = \|u\|^2_{H^1} - (\omega^2+1)\|u\|^2_{L^2}.$$

If ω^2 is not an eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions, then $\mathcal{A}(u,v) = 0$ for all $v \in H^1_0$ implies $u \equiv 0$. Thus, Helmholtz problem for ω^2 not an eigenvalue of $-\Delta$ has weak formulation with a bilinear form that satisfies the *inf-sup* condition.

Proof. Assume that (5) does not hold. This means that for any n we can find u_n such that

$$\sup_{v \in H^1_0} \frac{\mathcal{A}(u_n,v)}{\|v\|_{H^1_0}} \leq \frac{1}{n}\|u_n\|_{H^1_0}.$$

Obviously we can take $\|u_n\|_{H^1_0} = 1$. Thus, we have a sequence $\{u_n\}_{n=1}^\infty$, $\|u_n\|_{H^1} = 1$. Since $\{u_n\}$ is bounded in H^1 and the space H^1 is compactly embedded in L^2 we can extract a sub-sequence $\{u_k\}_{k=1}^\infty$ that converges in $L^2(\Omega)$ to its limit $u_0 \in L^2(\Omega)$. Then

$$0 \leq \sup_{v \in H^1_0} \frac{\mathcal{A}(u_n,v)}{\|v\|_{H^1_0}} = \sup_{v \in H^1_0, \|v\|_{H^1}=1} \mathcal{A}(u_n,v) \leq \frac{1}{n}.$$

This implies $0 \leq \mathcal{A}(u_n,v) \leq \frac{1}{n}$ for any $v \in H^1$ and $\mathcal{A}(u_n,u_n) \to 0$ when $n \to \infty$. Taking the limit in the first inequality we get

$$0 = \lim_{n\to\infty} \mathcal{A}(u_n,v) = \mathcal{A}(u_0,v) \quad \forall v \in H^1_0.$$

By assumption (b) $u_0 = 0$ as an element in $L^2(\Omega)$ and therefore $\|u_0\|_{L^2} = 0$. However, by Görding inequality (4) we have

$$\frac{1}{n} \geq \mathcal{A}(u_n,u_n) \geq \alpha\|u_n\|^2_{H^1} - \beta\|u_n\|^2_{L^2}.$$

By taking limit when $n \to \infty$ and using the following facts $\|u_n\|_{H^1} = 1$, and $\lim_{n\to\infty}\|u_n\|_{L^2} = 0$ we get an obvious contradiction

$$0 = \lim_{n\to\infty} \mathcal{A}(u_n,u_n) \geq \alpha > 0.$$

This completes the proof. □

Remark 2. *We remark that every second order elliptic problem falls into the class of probems covered by the above Theorem as long as the zero is not an eigenvalue of the corresponding operator.*

3. FEM for Indefinite Elliptic Problems

3.1. **Problem Formulation.** Now we consider the general second order elliptic problem

$$(V) \qquad \boxed{\text{find } u \in H_0^1(\Omega) \text{ such that } \mathcal{A}(u,v) = \mathcal{L}(v), \ \forall \ v \in H_0^1(\Omega) \ ,}$$

where

$$\mathcal{A}(u,v) = \int_\Omega \Big(K(x)\nabla u \cdot \nabla v - u\mathbf{b}(x) \cdot \nabla v + c(x)uv \Big)\, dx, \ \mathcal{L}(v) = \int_\Omega f v dx.$$

Recall that we have assumed that $K(x)$ is a symmetric uniformly in Ω positive definite matrix. Then it is easy to show that the bilinear form satisfies Görding inequality (this is left as an exercise). We assume that $\mathcal{A}(u,v) = 0$ for all $v \in H_0^1(\Omega)$ implies $u = 0$. Then according to Theorem 2 the problem (V) has unique solution.

Now we consider FEM for this problem using linear finite elements over a partition of the domain Ω into triangles. Let $\mathcal{V}_h \subset H_0^1(\Omega)$ be the corresponding finite element space. Then the FEM for the problem (V) is:

$$(V_h) \qquad \boxed{\text{find } u_h \in \mathcal{V}_h \text{ such that } \mathcal{A}(u_h,v) = \mathcal{L}(v), \ \forall \ v \in \mathcal{V}_h \ .}$$

The main question is whether this problem is well posed and if so, what is the error bound for the FE solution.

To simplify the exposition we note that $\mathcal{V}_h \subset H_0^1(\Omega) := \mathcal{V}$ and it inherits the norm in H_0^1 so that $\|u_h\|_\mathcal{V} = \|u_h\|_{H^1}$. Now we prove the following theorem:

Theorem 3. *If $h \le h_0$, where h_0 is sufficiently small (depending on Ω, coefficients of the differential equation, etc), then there is an independent of h constant α_0^*, such that the discrete inf-sup condition is valid, namely*

$$(6) \qquad \forall u_h \in \mathcal{V}_h : \qquad \sup_{v \in \mathcal{V}_h} \frac{\mathcal{A}(u_h, v)}{\|v\|_\mathcal{V}} \ge \alpha_0^* \|u_h\|_\mathcal{V}.$$

Proof. For the proof we shall need the following facts:

(1) the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous in $\mathcal{V}$, i.e. it satisfies the estimate
$$\mathcal{A}(v,w) \le C\|v\|_{H^1}\|w\|_{H^1}, \quad \forall v, w \in \mathcal{V};$$

(2) **Ritz**-projection $R_h v \in \mathcal{V}_h$ of a function $v \in \mathcal{V}$ defined as the solution to the FE problem $(\nabla R_h v, \nabla \phi) = (\nabla v, \nabla \phi)$, $\forall \phi \in \mathcal{V}_h$ is stable in H^1, that is
$$c_1 \|R_h v\|_{H^1} \le \|v\|_{H^1}, \quad \text{with} \quad c_1 > 0,$$
and approximates v in L^2-norm by satisfying
$$\|v - R_h v\|_{L^2} \le Ch\|v\|_{H^1}, \quad \text{with} \quad C > 0.$$

Now since $u_h \in H^1$ we use the *inf-sub* condition established in Theorem 2 an proceed as follows

$$\alpha_0 \|u_h\|_{H^1} \leq \sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, v)}{\|v\|_{H^1}} \leq \sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, v - R_h v)}{\|v\|_{H^1}} + \sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, R_h v)}{\|v\|_{H^1}}.$$

By using the definition of the Ritz projection, $(\nabla(R_h u - u), \nabla v) = 0$ for all $v \in \mathcal{V}_h$, the continuity of the bilinear form, and the approximation property of the Ritz projection, that is $\|R_h v - v\|_{L^2} \leq ch\|v\|_{H^1}$, for the first term we get

$$\begin{aligned}\sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, v - R_h v)}{\|v\|_{H^1}} &= \sup_{v \in H_0^1} \frac{-\omega^2(u_h, v - R_h v)}{\|v\|_{H^1}} \\ &\leq \sup_{v \in H_0^1} \frac{C\|u_h\|_{H^1}\|v - R_h v\|_{L^2}}{\|v\|_{H^1}} \leq Ch\|u_h\|_{H^1}.\end{aligned}$$

For the second term we use the stability of Ritz-projection to get

$$\sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, R_h v)}{\|v\|_{H^1}} \leq \sup_{v \in H_0^1} \frac{\mathcal{A}(u_h, R_h v)}{c_1\|R_h v\|_{H^1}} = \frac{1}{c_1} \sup_{v \in \mathcal{V}_h} \frac{\mathcal{A}(u_h, v)}{\|v\|_{H^1}}$$

As a result we get

$$(\alpha_0 - Ch)\|u_h\|_{H^1} \leq \frac{1}{c_1} \sup_{v \in \mathcal{V}_h} \frac{\mathcal{A}(u_h, v)}{\|v\|_{H^1}},$$

which produces the desired result for sufficiently small h. For example, if $h_0 = \frac{\alpha_0}{2C}$ we get $\alpha_0^* = \frac{c_1 \alpha_0}{2}$. □

Remark 3. *We have used the estimate* $\|v - R_h v\|_{L^2} \leq Ch\|v\|_{H^1}$*, which is valid if the solution of the elliptic problem* $(\nabla v, \nabla \phi) = (f, v)$ *has full regularity. This means that* $u \in H^2(\Omega)$*. This is the case of convex domain* Ω *with polygonal boundary. For non-convex domains however, the estimate is not valid. But in this case we can prove an estimate* $\|v - R_h v\|_{L^2} \leq Ch^{\frac{1}{2}}\|v\|_{H^1}$*, which will also lead to the desired inf-sup condition.*

As a result of this theorem we have the following result:

Theorem 4. *The solution of the problem* (V_h) *exists and satisfies the a priori estimate*

$$\|u_h\|_{H^1} \leq \frac{1}{\alpha_0^*}\|f\|_{L^2}.$$

3.2. **Error Analysis.** The error analysis follows in a standard way from Cea's Lemma. Now taking v to be the finite element interpolate $\Pi_h u \in \mathcal{V}_h$ and using the approximation properties of the interpolate,

$$\|u - \Pi_h u\|_{\mathcal{V}} \leq Ch\|u\|_{H^2} \tag{7}$$

we get the desired error estimate: $\|u_h - u\|_{H^1} \leq Ch\|u\|_{H^2}$.

In this lecture we used the approximation property (7) of the finite element interpolate of a function $u \in H^2(\Omega)$. In the next lecture we shall establish this error bound. More precisely, we shall prove the following theorem:

Theorem 5. *Assume the triangulation of Ω is shape-regular (we shall define this rigorously), the maximal diameter of the finite elements is h, and the finite element space consists of continuous piece-wise linear functions. Then for $u \in H^2(\Omega)$ we have*

$$\|u - \Pi_h u\|_{L^2} + h\|\nabla(u - \Pi_h u)\|_{L^2} \leq Ch^2 \|u\|_{H^2(\Omega)},$$

where $\Pi_h u$ is the nodal finite element interpolate.

4. Exercises

Problem 1: Consider the problem

$$\begin{aligned} -\Delta u + \mathbf{b} \cdot \nabla u + c(x)u &= f, \qquad && \text{on } \Omega, \\ u &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Here you may take Ω to be the unit square (in $\mathbf{R}^2$). Assume that $c(x) \geq 0$, the vector field $\mathbf{b}(x)$ is smooth and satisfies $\nabla \cdot \mathbf{b}(x) = 0$ and $f \in L^2(\Omega)$.

(1) Derive a variational formulation of the above problem. You must identify the space V, the bilinear form $\mathcal{A}(\cdot,\cdot)$, and the linear functional $\mathcal{L}(\cdot)$.
(2) Show that the resulting bilinear form is coercive.

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