The test will include definitions, derivation of simple properties, and one proof of important theoretical result.

**Problem 1.** (definitions and quick questions)
(a) Define positive definite $n \times n$ matrix $A$;
(b) When do we say that an $n \times n$ matrix $A$ is diagonally dominant?
(c) Show that strictly diagonally dominant matrix $A$ is nonsingular.
(d) Define the Euclidean norm of a vector $x \in \mathbb{R}^n$.
(e) Given the matrix $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, compute its $\| \cdot \|_\infty$-norm;
(f) Is this matrix non-singular?

**Problem 2.** (25 pts) Solve the following system of algebraic equations by Gauss elimination with backward substitution:
\[
\begin{align*}
x_1 - x_2 - x_3 &= 14 \\
x_2 + 4x_3 &= 0 \\
x_1 - 2x_2 + 2x_3 &= 0
\end{align*}
\]

**Problem 4.** Assume that one can perform the Gauss elimination for the system of linear equations $Ax = b$ without pivoting. Derive the corresponding formulas for the particular matrix below (called upper Hessenberg form and characterized by $a_{i,j} = 0$ for $i \geq j + 2$) and show that the forward elimination requires $\frac{n^2}{2} + O(n)$ multiplications:
\[
A = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \ldots & a_{1,k} & \ldots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \ldots & a_{2,k} & \ldots & a_{2,n-1} & a_{2,n} \\
0 & a_{3,2} & a_{3,3} & \ldots & a_{3,k} & \ldots & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & \ldots & a_{n,n-1} & a_{n,n}
\end{bmatrix}
\]

**Problem 5.** Problems 10, 11, 12 of Exercise set 6.3 of your book. Also Problems 5, 6, 7, 8, 11 of Exercise set 6.4 of your book.
Problem 6. Given the initial value problem \( y'(t) = f(t, y), \ y(t_0) = \alpha_0, \) where \( f(t, y) \) is a given function and \( \alpha_0 \) is a given constant. Consider the one-step method

\[
w_{i+1} = w_i + hf \left( t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i) \right), \quad w_0 = \alpha_0.
\]

(a) Estimate the local truncation error; (b) perform two steps with \( h = 0.1 \) for solving the initial value problem \( y' = 1 + \frac{y}{t}, \ 1 \leq t \leq 2, \ y(1) = 2. \)

Problem 7. Find the inverse of the matrix \( A \) by applying Gauss elimination to the extended matrix and check your result:

\[
A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 4 \end{bmatrix}
\]
to the extended matrix \( A_{\text{extended}} = \begin{bmatrix} 3 & -1 & 1 & : & 1 & 0 & 0 \\ 0 & -2 & -2 & : & 0 & 1 & 0 \\ 0 & 2 & 4 & : & 0 & 0 & 1 \end{bmatrix} \)

Problem 8. Problems 1a, 2a from Exercise set 6.5 of your book.

Problem 9. Problems 1a, 1,b, 2, 3a, 16, 17, 18, 19, 22, 23 from Exercise set 6.6 of your book.

Problem 10. Find the determinant and a norm of your choice of the matrix

\[
A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 4 \end{bmatrix}.
\]

Answer the following questions: (1) Is the matrix singular? (2) Is the matrix diagonally dominant? (c) Is the matrix strictly diagonally dominant?

Problem 11. Given an \( n \times n \) symmetric and positive definite matrix \( A \), derive the formulas for \( LL^t \)-factorization and find the leading term of the operation count for multiplication/divisions. Here \( L \) is a lower triangular matrix with positive diagonal elements.

Problem 12. Consider the IVP \( y' = f(t, y), \ t \in (a, b), \ y(a) = \alpha \) and its approximation by the following two-step explicit scheme:

\[
w_{i+1} = -4w_i + 5w_{i-1} + h(4f(t_i, w_i) + 2f(t_{i-1}, w_{i-1})), \ i = 1, 2, \ldots \ w_0 = \alpha, w_1 - \text{given}.
\]

(a) Estimate the local truncation error \( \tau_{i+1}(h) \);
(b) Show that this scheme is unstable;
(c) For \( f \equiv 0 \) find the solution if \( w_0 = 1 \) and \( w_1 = 2. \)

Problem 13. Consider the IVP \( y' = f(t, y), \ t \in (a, b), \ y(a) = \alpha \) and its approximation by the following two-step explicit scheme:

\[
w_{i+1} = w_{i-1} + 2hf(t_i, w_i), \ i = 1, 2, \ldots \ w_0 = \alpha, w_1 - \text{given}.
\]

(a) Estimate the local truncation error \( \tau_{i+1}(h) \);
(b) Show that this scheme is weakly stable;

Problem 14. (this is problem 8 of exercise set 5.10 of your book) Consider the IVP \( y' = 0, \ t \in (0, 5), \ y(0) = 0, \) which has exact solution \( y \equiv 0. \)

\[
w_{i+1} = 4w_i - 3w_{i-1}, \ i = 1, 2, \ldots \ w_0 = \alpha, w_1 = \epsilon, \ \epsilon \text{ is the round off error}.
\]
(a) Compute the solution \( w_2, w_3, w_4, w_5 \) to see how the error propagates.
(b) Find the solution \( w_i \) for all \( i \).

**Problem 15.** Consider the initial value problem \( y' = f(t, y) \), \( y(t_0) = y_0 \) and a corresponding two-step method

\[
 w_{i+1} = w_i + \frac{h}{2} \left\{ 3f(t_i, w_i) - f(t_{i-1}, w_{i-1}) \right\}, \quad i = 1, 2, ..., \quad w_0 = y_0, \; w_1 \text{ given.}
\]

(1) Estimate the local truncation error. Hint. Use that for \( y_j = y(t_j) \) one has \( f(t_j, y_j) = y_j' \).
(2) Find the roots of the characteristic polynomial and check whether this methods satisfy the root condition.

**Problem 16.** Given the Runge-Kutta method for the initial value problem \( y' = f(t, y) \) for \( a < t < b \) and \( y(a) = \alpha \) of the form

\[
 w_0 = \alpha, \quad w_{i+1} = w_i + \frac{h}{2} \left( f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i)) \right), \quad i = 0, 1, 2, ...
\]

Show that the local truncation error is \( O(h^2) \). Compute \( w_1, w_2 \) using this method with step-size \( h = 0.2 \) for the following problem

\[
 y' = 1 + \frac{t}{2}, \quad 1 \leq t \leq 2, \quad y(1) = 2.
\]

**Problem 17.** Consider the following multistep methods for the initial value problem \( y' = f(t, y) \) on the interval \((t_0, t_0 + d)\). Below \( t_i = t_0 + ih, \; h = d/N \), and \( w_i \) is an approximation to \( y(t_i) \):

\[
\begin{align*}
(1) \quad & w_{i+1} = \frac{4}{3} w_i - \frac{1}{3} w_{i-1} + \frac{2h}{3} f(t_{i+1}, w_{i+1}); \\
(2) \quad & w_{i+1} = w_{i-1} + 2hf(t_i, w_i); \\
(3) \quad & \text{Milne's method} \quad w_{i+1} = w_{i-3} + \frac{4h}{3} \left( 2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2}) \right).
\end{align*}
\]

We assume that the needed initial values \( w_0, \ldots, w_m \) for an \( m \)-step methods are generated by an one-step method. Relevant questions for any of these methods are: (1) estimate the local truncation error; (2) does the method (*) satisfy the root condition? (3) Is the method (*) strongly stable, weakly stable, or unstable?

**Problem 18.** Let \( A \) be an \( n \times n \) matrix. Derive an algorithm for computing the \( LU \)-factorization where \( L \) is a lower triangular matrix with 1-s on the main diagonal and \( U \) is an upper triangular matrix. Derive a formula for count of long operations (multiplications, divisions) need to compute \( L \) and \( U \).

**Problem 19.** Find the inverse of the matrix using the procedures involving the augmented matrix:

\[
\mathbb{L} : \mathbb{I} = \begin{bmatrix}
 l_{1,1} & 0 & 0 & \cdots & 0 & : & 1 & 0 & 0 & \cdots & 0 \\
l_{2,1} & 1 & 0 & \cdots & 0 & : & 0 & 1 & 0 & \cdots & 0 \\
l_{2,1} & 0 & 1 & \cdots & 0 & : & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
l_{n,1} & 0 & 0 & \cdots & 1 & : & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]
or similar case: find the inverse of the matrix using the procedures involving the augmented matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & : & 1 & 0 & 0 & \ldots & 0 \\
0 & l_{22} & 0 & \ldots & 0 & : & 0 & 1 & 0 & \ldots & 0 \\
0 & l_{32} & 1 & \ldots & 0 & : & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & l_{n2} & 0 & \ldots & 1 & : & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

More challenging problems

**Problem X.** Take the method for finding the inverse of a given \(n \times n\) matrix \(A = \{a_{i,j}\}\) by straightforward Gauss (or Jordan) elimination (Problem 7 is a particular case for \(n = 3\)). First you write down the augmented matrix \(A\) and apply the Gauss process to this as discussed in class:

\[
A = \begin{pmatrix}
    a_{1,1} & a_{1,2} & \ldots & a_{1,n} & : & 1 & 0 & \ldots & 0 \\
    a_{2,1} & a_{2,2} & \ldots & a_{2,n} & : & 0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{n,1} & a_{n,2} & \ldots & a_{n,n} & : & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

1. Derive the Gauss elimination algorithm without pivoting for the augmented matrix (in terms of a triple loop);
2. Show that the count of multiplications/divisions is \(\frac{4}{3}n^3 + O(n^2)\);
3. Derive the Jordan elimination algorithm without pivoting for the augmented matrix (in terms of a triple loop);
4. Show that the count of multiplications/divisions is \(\frac{3}{2}n^3 + O(n^2)\);

**Problem XX.** Show that the matrices \(A\) and \(B\) are positive definite:

\[
A = \begin{pmatrix}
    2 & -1 & 0 & \ldots & 0 & 0 \\
   -1 & 2 & -1 & \ldots & 0 & 0 \\
    0 & -1 & 2 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & 2 & -1 \\
    0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix} \quad B = \begin{pmatrix}
    2 & 1 & 0 & \ldots & 0 & 0 \\
    1 & 2 & 1 & \ldots & 0 & 0 \\
    0 & 1 & 2 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & 2 & 1 \\
    0 & 0 & 0 & \ldots & 1 & 2
\end{pmatrix}
\]