ON $L_p$-APPROXIMATION OF FUNCTIONS WHOSE $m$th DERIVATIVE IS OF BOUNDED VARIATION

by

RONALD DEVORE

1. Introduction. Let $f$ be an element of $L_p[-1, 1]$, $1 \leq p \leq \infty$, and $S$ and $T$ be subsets of $L_p[-1, 1]$. Then $E_p(f, T) = \inf_{g \in T} \|f - g\|_p$ is the error in approximating $f$ by elements of $T$ in the $L_p[-1, 1]$ norm and $E_p(S, T) = \sup_{f \in S} E_p(f, T)$ is the error in approximating elements of $S$ by elements of $T$ in the $L_p[-1, 1]$ norm.

A problem of particular interest is the determination of $E_p(S, T)$ when $S$ is characterized by some structural property of its elements and $T$ is one of the classes $P_n$ of algebraic polynomials of degree $\leq n$ or $T_n$ of trigonometric polynomials of degree $\leq n$. The first result of this type is the classical result of J. Favard [1] which for the interval $[-\pi, \pi]$ instead of $[-1, 1]$ can be stated as follows:

If $W_m$ is the class of those $2\pi$ periodic functions $f$ for which $f^{(m-1)}$ is absolutely continuous and $|f^{(m)}(x)| \leq 1$ a.e., then

$$E_m(W_m, T_n) = \frac{K_m}{n^m},$$

where

$$K_m = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^{m+1}j}{(2j+1)^m}.$$

Another result in this direction, for $L_1$ approximation, was obtained by S. Nikolski [2]. Let $A_m$ be the class of those functions $f$ for which $f^{(m-1)}$ is absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is equivalent to a function $g$ whose total variation on $[-1, 1]$ is $\leq 1$. Then for $n \geq m$, $m = 1, 2, ...$

(1.1)

$$E_1(A_m, P_n) = \frac{M_1(m, P_n)}{m!} E_1(x_n^m, P_n)$$

where

(1.2)

$$(x-a)^m = \begin{cases} (x-a)^m & \text{for } x > a \\ 0 & \text{for } x \leq a \end{cases}$$

and

(1.3)

$$M_p(m, P_n) = \sup_{|x| \leq 1} \frac{E_p((x-a)^m, P_n)}{E_p(x_n^m, P_n)}$$

Nikolski [3] has also studied the functions $M_1(m, P_n)$ and $E_1(x_n^m, P_n)$. He has determined $E_1(x_n^m, P_n)$ explicitly when $m$ is an odd positive integer and has proved the following asymptotic formulae.

(1.4)

$$E_1(x_n^m, P_n) = \frac{c_m}{n^{m+2}} + O\left(\frac{\log n}{n^{m+2}}\right) \quad (n \to \infty)$$

(1.5)

$$M_1(m, P_n) = 1 + O\left(\frac{\log n}{n}\right) \quad (n \to \infty).$$
In this paper, we shall study the $L_p$ approximation ($1 \leq p \leq \infty$) of the class $A_m$ by $P_n$. First of all, we shall show that the result of Nikol'ski (1.1) is valid for $1 \leq p \leq \infty$. More precisely, in Theorem 1 we show that for $1 \leq p \leq \infty$ and $n \geq m$, $m=1, 2, \ldots$

$$E_p(A_m, P_n) = \frac{M_p(m, P_n)}{m!} E_p(x^m_+, P_n).$$

Next, we shall solve explicitly the problem of best $L_1$ approximation to the functions $x^m_+, m=1, 2, \ldots$, on the interval $[-1, 1]$ by algebraic polynomials by means of more general results which are interesting in themselves. In Theorem 2, we shall determine the polynomials of best $L_1$ approximation on $[-1, 1]$ of degree $\leq 2n - 2, 2n - 1$ to even functions of the form $h(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

A similar result for odd functions is Theorem 3 which determines the polynomials of best $L_1$ approximation on $[-1, 1]$ of degree $\leq 2n - 1, 2n$ to functions of the form $xh(x^2)$ with $h^{(n)}$ of constant sign on $(0, 1)$.

Finally, using the constructive method employed in the proof of Theorem 1 and the explicit determination of polynomials of best $L_1$ approximation to the functions $x^m_+$, given by Theorems 2 and 3, we shall show that for each function $f$ in $A_m$ there is a polynomial

$$P_n(x) = A_0(n) + A_1(n)x + \ldots + A_n(n)x^n$$

satisfying:

1° $\|f - P_n\|_1 \leq \frac{C_1}{n^{m+1}}$

2° $|A_k(n)| \leq C_2 3^n$ \quad $k = 0, 1, \ldots, n$

where $C_1$ and $C_2$ depend only on $m$.

The same result was obtained by J. Korevaar [4] for the class $K_m$ of those functions $f$ for which $f^{(m-1)}$ is absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is continuous except for a finite number of jump discontinuities on $[-1, 1]$.

2.1. Approximation of $A_m$ by $P_n$ in the $L_p$ norm.

**Theorem 1.** If $1 \leq p \leq \infty$, then for $n \geq m$, $m=1, 2, \ldots$

$$E_p(A_m, P_n) = \frac{M_p(m, P_n)}{m!} E_p(x^m_+, P_n).$$

**Proof.** Let $f \in A_m$ and $g=f^{(m)}$ a.e. where the variation of $g$ on $[-1, 1]$ is $\leq 1$. If $\varepsilon > 0$, there is a partition $-1 = x_0 < x_1 < \ldots < x_r = 1$ for which

$$\int_{-1}^{1} |g(x) - I_0(x)| \, dx \leq \frac{\varepsilon}{2^{p(m-1)}}$$

---

* The asymptotic behaviour of $M_p(m, P_n)$ and $E_p(x^m_+, P_n)$ has been given by Rassin in *Dokl. Trans. A. M. S.* 6 (1965) 1171—1174.

*Studia Scientiarum Mathematicarum Hungarica* 3 (1968) 348
where \( l_0 \) is the step function which has the value \( g(x_k) \) on \( (x_k, x_{k+1}] \). We have

\[
l_0(x) = g(x_0) + \sum_{k=1}^{r} (g(x_k) - g(x_{k-1}))(x - x_k)_+^0.
\]

We recursively define for \( j = 1, 2, \ldots, m \)

\[
l_j(x) = \int_{-1}^{x} l_{j-1}(t) \, dt + f^{(m-j)}(-1).
\]

Then for \( x \in [-1, 1] \), we have

\[
|f^{(m-j)}(x) - l_j(x)| = \left| \int_{-1}^{x} (f^{(m-j+1)}(t) - l_{j-1}(t)) \, dt \right| \leq \varepsilon
\]

\[
\leq \int_{-1}^{1} |f^{(m-j+1)}(t) - l_{j-1}(t)| \, dt \leq \frac{\varepsilon}{2^{m-j}}
\]

for \( j = 1, 2, \ldots, m \). Thus, \( \|f - l_m\|_p \leq \varepsilon \).

We have

\[
(2.1) \quad l_m(x) = P(x) + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1}))(x - x_k)_+^m
\]

where \( P \) is a polynomial of degree \( \leq m \). Let \( P_k \in P_n \) satisfy

\[
\|(x - x_k)_+^m - P_k(x)\|_p = E_p((x - x_k)_+^m, P_n).
\]

Then for \( Q = P + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) \, P_k \), we have \( Q \in P_n \) and

\[
\|l_m - Q\|_p \leq \frac{1}{m!} \sum_{k=1}^{r} |g(x_k) - g(x_{k-1})| \| (x - x_k)_+^m - P_k(x) \|_p
\]

and so

\[
\|l_m - Q\|_p \leq \frac{M_p(m, P_n)}{m!} E_p(x_+^m, P_n).
\]

Thus,

\[
\|f - Q\|_p \leq \|f - l_m\|_p + \|l_m - Q\|_p \leq \varepsilon + \frac{M_p(m, P_n)}{m!} E_p(x_+^m, P_n).
\]

Since \( \varepsilon \) is arbitrarily small and \( f \) is any function in \( A_m \), we have

\[
E_p(A_m, P_n) \leq \frac{M_p(m, P_n)}{m!} E_p(x_+^m, P_n).
\]

The functions \( \frac{(x - a)_+^m}{m!} \) are in \( A_m \) and

\[
E_p\left( \frac{(x - a)_+^m}{m!}, P_n \right) = \frac{E_p((x - a)_+^m, P_n)}{m!}.
\]
Therefore,

\[ E_p(A_m, P_n) = \frac{M_p(m, P_n)}{m!} E_p(x_m^m, P_n) \]

and the theorem is proved.

2. 2. \(L_1\) approximation. We now establish two theorems on \(L_1\) approximation which are of interest in themselves and give in particular the value of \(E_1(x_m^m, P_n)\). We denote by \(L(f, x_1, x_2, \ldots, x_n, x)\) the Lagrange interpolation polynomial which interpolates the function \(f\) at the points \(x_1, x_2, \ldots, x_n\).

**Theorem 2.** Let \(f(x) = h(x^2)\) where \(h^{(n)}(x) > 0 (h^{(n)}(x) < 0)\) on \((0, 1)\). Then the polynomial of best \(L_1\) approximation to \(f\) on \([-1, 1]\) of degree \(\equiv 2n - 2, 2n - 1\) is \(L(f, t_1, t_2, \ldots, t_{2n}^2, x)\) where

\[ t_k = -\cos \left( \frac{k\pi}{2n+1} \right) \quad k = 1, 2, \ldots, 2n. \]

Also,

\[ E_1(f, P_{2n-2}) = E_1(f, P_{2n-1}) = \left| \int_{-1}^{1} f(x) \text{ sgn} \ U_{2n} \, dx \right| \]

where \(U_{2n}\) is the Čebyšev polynomial of the second kind.

**Theorem 3.** Let \(f(x) = xh(x^2)\) where \(h^{(n)}(x) > 0 (h^{(n)}(x) < 0)\) on \((0, 1)\). Then the polynomial of best \(L_1\) approximation to \(f\) on \([-1, 1]\) of degree \(\equiv 2n - 1, 2n\) is \(L(f, t_1, t_2, \ldots, t_{2n+1}^2, x)\) where

\[ t_k = -\cos \left( \frac{k\pi}{2n+1} \right) \quad k = 1, 2, \ldots, 2n+1. \]

Also,

\[ E_1(f, P_{2n-1}) = E_1(f, P_{2n}) = \left| \int_{-1}^{1} f(x) \text{ sgn} \ U_{2n+1} \, dx \right|. \]

**Proofs.** The proofs are similar and only that of Theorem 2 will be given. From the theorem of S. N. Bernstein [5, p.p. 330–332], it is sufficient to show that \(f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)\) changes sign at \(t_1, t_2, \ldots, t_{2n}\) and only these points on \([-1, 1]\). Let \(Q(x) = L(h, t_1^2, t_2^2, \ldots, t_{n}^2, x)\). Then the degree of \(Q\) is \(\equiv n - 1\) and from Cauchy's remainder formula for Lagrange interpolation we have that for each \(x \in (0, 1)\) there is a \(\xi_x \in (0, 1)\) such that

\[ h(x) - Q(x) = \frac{h^{(n)}(\xi_x)}{n!} (x - t_1^2) \ldots (x - t_n^2). \]

So that,

\[ f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x) = h(x^2) - Q(x^2) = \frac{h^{(n)}(\xi_x^2)}{n!} (x - t_1) \ldots (x - t_{2n}) \]

for \(x \in (-1, 1)\) \(x \neq 0\). Thus \(f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)\) changes sign at \(t_1, t_2, \ldots, t_{2n}\) and only these points on \([-1, 1] \setminus \{0\}\). Since the function \(f(x) - L(f, t_1, t_2, \ldots, t_{2n}, x)\) is even, it does not change sign at 0. Finally, since the degree of \(Q(x^2)\) is \(2n - 2\), the theorem is proved.
If we consider the function \( f(x) = |x|^s \) \((s > -1)\), Theorem 2 gives the following corollary which was proved by Nikolski using a different method based on Descartes' rule of signs.

**Corollary 1.** The polynomial of best \( L_1 \) approximation to \(|x|^s\) \((s > -1)\) on \([-1, 1]\) of degree \( \leq 2n-2 \), \(2n-1\) is \( L(|x|^s, t_1, t_2, \ldots, t_{2n}, x) \) where
\[
t_k = -\cos \left( \frac{k\pi}{2n+1} \right) \quad k = 1, 2, \ldots, 2n.
\]

Also,
\[
E_1(|x|^s, P_{2n-2}) = E_1(|x|^s, P_{2n-1}) = \frac{2}{s+1} \left| 2 \sum_{k=1}^{n} (-1)^k \left( \cos \left( \frac{k\pi}{2n+1} \right) \right)^{s+1} + 1 \right|.
\]

Let us now consider the function \( f(x) = x^{-m-1}|x| \) when \( m \) is an integer \( \geq -1 \). Since \( x_{+}^m = \frac{1}{2} (|x|x^{m-1} + x^m) \), we have \( E_1(x_{+}^m, P_n) = \frac{1}{2} E_1(f, P_n) \) for \( n \equiv m \). Therefore, we have the following two corollaries to Theorems 2 and 3.

**Corollary 2.** For \( m \) an odd positive integer and \( n \equiv m \)
\[
E_1(x_{+}^m, P_n) = \frac{1}{m+1} \left| 1 + 2 \sum_{k=1}^{\frac{n+1}{2}} (-1)^k \left( \cos \left( \frac{k\pi}{2\frac{n}{2}+1} \right) \right)^{m+1} \right|.
\]

**Corollary 3.** For \( m \) an even non-negative integer and \( n \equiv m \)
\[
E_1(x_{+}^m, P_n) = \frac{1}{m+1} \left| 1 + 2 \sum_{k=1}^{\frac{n+1}{2}} (-1)^k \left( \cos \left( \frac{k\pi}{2\frac{n}{2}+1} \right) \right)^{m+1} \right|.
\]

Nikolski [3] has shown that \( M_1(m, P_n) = 1 + O\left(\log\frac{n}{n}\right) \). Thus in the case \( p = 1 \), Theorem 1 becomes:

**Corollary 4.** For \( n \equiv m, m = 1, 2, \ldots \)
\[
E_1(A_m, P_n) = \left( 1 + O\left(\log\frac{n}{n}\right) \right) \frac{E_1(x_{+}^m, P_n)}{m!}
\]
where \( E_1(x_{+}^m, P_n) \) is given in corollaries 2 and 3.

3. Estimates on the coefficients of polynomial approximations to functions in \( A_m \)

The principal result of this section is the following theorem.

**Theorem 4.** If \( f \in A_m \), there is a polynomial
\[
P_n(x) = A_0(n) + A_1(n)x + \ldots + A_n(n)x^n
\]
satisfying
\[
1^\circ \quad \int_{-1}^{1} |f(x) - P_n(x)| \, dx \equiv \frac{C_1}{n^{m+1}}
\]
\[
2^\circ \quad |A_k(n)| \equiv C_2 3^n \quad k = 0, 1, \ldots, n
\]
where \( C_1 \) and \( C_2 \) depend only on \( m \).
PROOF. We consider only the case when both \( m \) and \( n \) are odd. Other cases are handled in a similar manner. We can also assume that \( n \leq m \). Let \( Q_n \) denote the polynomial of best \( L_1 \) approximation to \( x^n \) on \([-1, 1]\) of degree \( \leq n \) which is given by Theorem 2. Then for \( |a| \leq 1 \)

\[
(3.1) \quad \int_{-1}^{1} \left| (x-a)^m + 2m Q_n \left( \frac{x-a}{2} \right) \right| \, dx = \int_{-1-a}^{1-a} \left| x^n + 2m Q_n \left( \frac{x}{2} \right) \right| \, dx \leq
\]

\[
\leq \int_{-2}^{2} \left| x^n + 2m Q_n \left( \frac{x}{2} \right) \right| \, dx = 2^{m+1} E_1 (x^n_+, P_n).
\]

Let \( f \in A_m \). Using the notation introduced in the proof of Theorem 1, for a suitable partition \(-1 = x_0 < x_1 < \ldots < x_r = 1\), we have

\[
(3.2) \quad \int_{-1}^{1} |f(x) - l_m(x)| \, dx \leq \varepsilon \leq E_1 (x^n_+, P_n).
\]

We define

\[
(3.3) \quad P_n(x) = P(x) + \frac{1}{m!} \sum_{k=1}^{r} \left( g(x_k) - g(x_{k-1}) \right) 2m Q_n \left( \frac{x-x_k}{2} \right) =
\]

\[
= A_0(n) + A_1(n)x + \ldots + A_n(n)x^n.
\]

Using (2.1), (3.1), and (3.3), we have

\[
(3.4) \quad \int_{-1}^{1} |l_m(x) - P_n(x)| \, dx \leq
\]

\[
\leq \frac{1}{m!} \sum_{k=1}^{r} \left| g(x_k) - g(x_{k-1}) \right| \int_{-1}^{1} \left| (x-x_k)^m + 2m Q_n \left( \frac{x-x_k}{2} \right) \right| \, dx \leq \frac{2^{m+1}}{m!} E_1 (x^n_+, P_n).
\]

Thus, by (3.2), (3.4), and (1.4)

\[
\int_{-1}^{1} |f(x) - P_n(x)| \, dx \leq \left( \frac{2m}{m!} + 1 \right) E_1 (x^n_+, P_n) \leq \frac{C_1}{n^{m+1}}.
\]

We now estimate the coefficients of \( P_n \). From Theorem 2 and the Lagrange interpolation formula, it follows that

\[
Q_n \left( \frac{x-a}{2} \right) = L \left( x^n_+, t_1, t_2, \ldots, t_{n+1}, \frac{x-a}{2} \right) = \sum_{t_k > 0} \frac{t_k^n U_{n+1}(x-a)}{U_{n+1}(t_k) \left( \frac{x-a}{2} - t_k \right)}.
\]
i.e.

\begin{equation}
Q_{n}\left(\frac{x-a}{2}\right) = \sum_{k=\frac{n+1}{2}+1}^{n+1} (-1)^{k+1}(n+2) \left( \frac{-\cos\left(\frac{k\pi}{n+2}\right)}{\sin^{2}\left(\frac{k\pi}{n+2}\right)} \right)^{m} U_{n+1}\left(\frac{x-a}{2}\right) = B_{0}(n,a) + B_{1}(n,a)x + \ldots + B_{n}(n,a)x^{n}.
\end{equation}

Thus, by Cauchy's inequality and the maximum modulus principle we have for |a| \leq 1

\begin{equation}
|B_{k}(n,a)| \leq \sup_{|z|=1} \left| Q_{n}\left(\frac{z-a}{2}\right) \right| \leq \sup_{|z|=1} \left| Q_{n}(z) \right| \leq \left\{ \frac{(n+2)^{2}}{\sin^{2}\left(\frac{\pi}{n+2}\right)} \right\} \max_{k} \left\{ \sup_{|z|=1} \left| U_{n+1}(z) \right| \right\}.
\end{equation}

Since

\begin{equation}
\sin^{-2}\left(\frac{\pi}{n+2}\right) \leq \frac{1}{4} (n+2)^{2} \quad \text{and for} \quad |z| = 1 \quad |z-t_{k}| \leq 1 - |t_{k}| \leq 1 - \cos\left(\frac{\pi}{n+2}\right) \leq \frac{2}{(n+2)^{2}},
\end{equation}

we have for |a| \leq 1 and k = 0, 1, \ldots, n

\begin{equation}
|B_{k}(n,a)| \leq \frac{(n+2)^{6}}{16} \sup_{|z|=1} \left| U_{n+1}(z) \right|.
\end{equation}

The leading coefficient of \( U_{n+1} \) is 2\( n+1 \) and thus

\begin{equation}
\sup_{|z|=1} |U_{n+1}(z)| = 2^{n+1} \sup_{|z|=1} |(z-t_{1})(z-t_{2}) \ldots (z-t_{n+1})| = 2^{n+1} \sup_{|z|=1} \left| (z^{2} - t_{1}^{2})(z^{2} - t_{2}^{2}) \ldots \left( z^{2} - t_{n+1}^{2} \right) \right| \leq 2^{n+1}(1+t_{1}^{2})(1+t_{2}^{2}) \ldots \left( 1+t_{n+1}^{2} \right).
\end{equation}

Since

\begin{equation}
(1+t_{1}^{2})(1+t_{2}^{2}) \ldots \left( 1+t_{n+1}^{2} \right) \leq \exp\left( \sum_{k=1}^{n+1} \cos^{2}\left( \frac{k\pi}{n+2} \right) \right) = \exp\left( \frac{n+2}{4} \right)
\end{equation}

it follows that

\begin{equation}
\sup_{|z|=1} |U_{n+1}(z)| \leq 2^{n+1} \exp\left( \frac{1}{4} (n+2) \right).
\end{equation}

Using (3.6), we find finally, that

\begin{equation}
|B_{k}(n,a)| \leq (n+2)^{6} 2^{n} \exp\left( \frac{n+2}{4} \right) \leq C_{0} 3^{n}
\end{equation}

where \( C_{0} \) is a constant independent of \( n \), and \( k = 0, 1, \ldots, n \).
Next, from (3.3) and (3.5), it follows that

\[ A_j(n) = a_j + \frac{1}{m!} \sum_{k=1}^{r} (g(x_k) - g(x_{k-1})) 2^m B_j(n, a) \]

where \( a_j \) is the \( j \)th coefficient of \( P \). Hence, from (3.7) it follows that

\[ |A_j(n)| \leq |a_j| + \frac{2^m}{m!} \sum_{k=1}^{r} |g(x_k) - g(x_{k-1})| C_0 3^n \leq |a_j| + C_0 3^n \leq C_2 3^n \]

for \( j = 0, 1, \ldots, n \) and the theorem is proved.

REFERENCES


Oakland University Rochester, Michigan, USA

(Received June 8, 1967.)