MAXIMAL FUNCTIONS AND THEIR APPLICATION TO RATIONAL APPROXIMATION

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ABSTRACT. The Hardy Littlewood maximal functions as well as other maximal functions are used to give simple constructive proofs of the results of Popov and Brudnyi on rational approximation.

1. INTRODUCTION. Maximal functions are important tools in various areas of analysis, most notably in differentiation theory and the study of mapping properties of operators. As we shall see, such maximal functions can also be used in a fundamental way to derive results on the approximation by rational functions. Specifically, we shall show how to give simple, constructive proofs of results of V. Popov [8,9] and Yu. Brudnyi [1]. These results center on various smoothness conditions on a function $f$ which guarantee that

$$\begin{align*}
\sup_n r_n(f) &= O(n^{-\alpha}) \\
\text{where} \\
\inf_{\deg R = n} \frac{R(f)}{q} &= 1
\end{align*}$$

These norms in this paper are on $[0,1]$ unless otherwise indicated.

Popov gave the famous inequality

$$r_n(f)_{\infty} \leq c f^{1+}\frac{1}{\text{BV}} n^{-2}$$

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This estimate should be contrasted with the approximation by polynomials
where one can only prove the error is $O(n^{-1})$ with $n$ the degree of the
approximating polynomials (consider for example $f(x) = |x|$ on $[-1,1]$)
[6 p. 94]). It follows in a simple way from (1.3) (as was noted by C. Freud
[5]) that for each $f \in \text{Lip} 1$

\begin{equation}
    r_n(f) = o_f(n^{-1}).
\end{equation}

This is a positive solution to the famous conjecture of D.J. Newman [7].

Maximal functions are used here to give local error estimates for the
approximation by polynomials. For example, it follows from the remainder
formula for interpolation by the Taylor polynomial $P_xf$ of degree $k-1$ that

\begin{equation}
    |f(y) - P_xf(y)| \leq |y-x|^{k-1} \int_x^y |f^{(k)}(t)| dt \leq |y-x|^k \inf_{x \leq u < y} H(f^{(k)})(u)
\end{equation}

where $H$ is the Hardy Littlewood maximal function:

\begin{equation}
    Hg(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |g|
\end{equation}

with the sup taken over all intervals $I \subset [0,1]$ which contain $x$.

One exploits (1.5) in rational approximations by choosing $n$ intervals
$I_1, \ldots, I_n$ where $\inf_I H f^{(k)}$ are approximately equal and taking $x = \xi_I
with $\xi_I \in I_I$. The local polynomials $P_{\xi_I}$ are then pieced together using a
simple partition of unity with low degree rational functions to give a
rational approximation to $f$. This technique is illustrated in its simplest
form in [52] where we prove Popov's result (1.3) by using (1.5) with $k=2$.

The more general results of Brudnyi require extensions of (1.5) to
estimate $Hf - P_x f \in L_q(I)$ in terms of the $\alpha$-th order smoothness of $f$ in $L_p$.

There are two essential difficulties that prevents the use of simple maximal
operators. First, $\alpha$ is not necessarily an integer and second and more
importantly $p$ is in general $< 1$. The usual notion of distributional
derivative does not apply to $p < 1$. Fortunately there are maximal functions
which will replace the role of $H(f^{(k)})$ in (1.5) for all $\alpha, p > 0$. These
were introduced by A.P. Calderon [2] and studied extensively in [4]. With
these maximal functions, one can show the existence of a polynomial $P_x f$ (now
based on Peano derivatives of \( f \) which generalizes (1.5) to all \( n, p > 0 \).

In §4, we show that any function \( f \) of smoothness \( \alpha \) in \( L^p \) (see §3 for the precise meaning of this) with \( p > \alpha \): \( (\alpha + \frac{1}{q})^{-1} \) satisfies

\[
(1.7) \quad r_n(f)_{q} = O(n^{-\alpha}).
\]

The index \( \sigma \) is the smallest index which guarantees that functions with \( \sigma \) order smoothness in \( L^p \) are in \( L^q \). This same index occurs in optimal knot spline approximation (our techniques apply to this problem as well).

The estimate (1.7) is a slight variant of the results announced by Brudnyi [1]. He uses Besov spaces in his description of smoothness. We should note that we have not seen the proofs of Brudnyi's results, however he does state in [3, p. 320] that his proofs are not constructive.

2. POPOV'S THEOREM. Suppose \( I_1, \ldots, I_m \) are disjoint intervals in \([0, 1]\) with \( \bigcup_{j=1}^{m} I_j = [0, 1] \) and \( \xi_j \in I_j, j=1, \ldots, m \). Let

\[
(2.1) \quad \phi_j(y) = (1+|I_j|^{-2}(y-\xi_j)^2)^{-2}, \quad j=1, \ldots, m.
\]

Then,

\[
(2.2) \quad \phi_j(y) \geq 2^{-2}, \quad y \in I_j; \quad j=1, \ldots, m.
\]

Let \( \phi = \sum_{j=1}^{m} \phi_j \), so that \( \phi \geq 2^{-2} \) on \([0, 1]\). The rational functions

\[
(2.3) \quad R_j = \phi_j/\phi
\]

satisfy

\[
(2.4) \quad \begin{aligned}
1) & \sum_{j=1}^{m} R_j \equiv 1 \\
11) & R_j(y) \leq 4(1+|I_j|^{-2}(y-\xi_j)^2)^{-2} \quad y \in [0, 1].
\end{aligned}
\]

**Theorem 2.1.** If \( f' \in BV \), there is a rational function \( R \) of degree \( \leq n \) such that

\[
(2.5) \quad |f - R|_{\infty} \leq c \|f'\|_{BV} n^{-2}
\]

with \( c \) an absolute constant.
PROOF. Since any function $f$ with $f' \in BV$ can be approximated uniformly by functions $g$ with $\|g\|_1 \leq \|f'\|_{BV}$, it will be enough to prove (2.5) with $\|f'\|_{BV}$ replaced by $\|f\|_1$. Also it is enough to construct $R$ of degree $\leq 16n$.

Now, the Hardy-Littlewood maximal function maps $L_1$ boundedly into $L_r$ for all $r < 1$. This follows from the fact that $M$ is of weak type $(1,1)$ [10, p.5]. If we take $r = 3/4$ (actually any $r$ strictly between $1/2$ and $1$ will do), then

$$\|M(f^*)\|_r \leq c_0 \|f\|_1.$$ 

It will be enough to establish our result for $f$ with $\|f\|_1 = c_0^{-1}$. For such an $f$, choose intervals $I_1, \ldots, I_{2n}$ such that

\begin{enumerate}
  \item[(2.6)] 
    \begin{enumerate}
      \item[i)] the $I_j$ have pairwise disjoint interiors and $[0,1] = \bigcup_{j=1}^{2n} I_j$
      \item[ii)] $|I_j| \leq n^{-1}$, $j=1, \ldots, 2n$
      \item[iii)] $\int_{I_j} [M(f^*)]|^{1/r} \leq n^{-1}$, $j=1, \ldots, 2n$.
    \end{enumerate}
\end{enumerate}

These intervals can be gotten by first finding $n$ intervals which satisfy i) and iii) and then further subdividing them so as to guarantee ii).

Choose $\xi_j \in I_j$ so that $M(f^*)(\xi_j) = \inf_{I_j} M(f^*)$. The inf is attained since $M(f^*)$ is lower semicontinuous. Let $R_j$ be the partition of unity (2.3) for this choice of $I_j$ and $\xi_j$. If $P_{\xi_j}(y) = f(\xi_j) + f'((\xi_j)(y-\xi_j)$, then (1.5) holds with $k=2$ and $x=\xi_j$, $j=1, \ldots, 2n$. Set

$$R := \sum_{j=1}^{2n} P_{\xi_j} R_j.$$ 

Since $R_j = \phi_j/\phi$ with $\phi_j$ of degree $\leq 4$ and $\phi$ of degree $\leq 8n$, we have $\deg R \leq 16n$.

Now, we estimate $f-R$ using (1.5) and (2.4). First observe that from (1.5) and (2.6) iii), we have

\begin{align*}
|f(y) - P_{\xi_j}(y)| &\leq (y-\xi_j)^2 M(f^*)(\xi_j) = (y-\xi_j)^2 \inf_{I_j} M(f^*) \\
&\leq (y-\xi_j)^2 \left( \frac{1}{|I_j|} \int_{I_j} [M(f^*)]|^{1/r} \right)^{1/r} \leq (y-\xi_j)^2 \left( n|I_j| \right)^{-1/r}.
\end{align*}
Using this together with (2.4) ii), we have

\[(2.8) \quad |f(y)-R(y)| \leq \frac{2n}{1} |f(y)-p_{E_j}(y)||e_j(y)| \]

\[\leq 4^{2n} \frac{1}{1} (y-E_j)^2 |n| |I_j| + (1+|I_j|)^{-2} (y-E_j)^2 \leq \]

\[\leq 4 \frac{1}{1} S_{v}(y) \]

where \(S_{v}(y)\) is the sum of those terms for intervals \(I_j\) which satisfy

\[2^{-v} \leq |I_j| \leq 2^{-v+1} \quad \text{(recall $|I_j| \leq n^{-1}$ because of (2.6) ii))}.\]

Since \((y-E_j)^2 \leq |I_j|^2 (1+|I_j|)^{-2} (y-E_j)^2\), we have

\[(2.9) \quad S_{v}(y) \leq 4(2^{v+1})^{1/2} 2^{v-2} \sum_{I_j \in \mathcal{I}_v} (1+|I_j|)^{-2} (y-E_j)^{-2} \]

with \(\mathcal{I}_v\) the set of those \(I_j\) which appear in \(S_{v}\). Each \(I_j \in \mathcal{I}_v\) has

length \(\geq 2^{-v} \leq n^{-1}\) and the \(I_j\) are disjoint. Therefore for any integer \(k \geq 0\)

there are at most four \(E_j\) with \(k 2^{-v} \leq |y-E_j| < (k+1) 2^{-v} \leq n^{-1}\). Using

this in (2.9), we have

\[S_{v}(y) \leq 16n^{-2} 2^{-v(2-1/r)} \sum_{k=0}^{\infty} (1+(k/2)^2)^{-1} \leq cn^{-2} 2^{-v(2-1/r)} \]

Thus, (2.8) gives

\[|f(y)-R(y)| \leq cn^{-2} \frac{1}{1} 2^{-v(2-1/r)} \leq cn^{-2}. \]

3. MAXIMAL FUNCTIONS. Fix \(a, p > 0\). If \(I\) is an interval let \(P_{I}f\) denote

a best approximation to \(f\) from polynomials of degree \((a)\) (greatest integer

strictly less than \(a\)) in \(L_{p}(I)\). We define

\[(3.1) \quad f_{a,p}^{b}(x) := \sup_{x \in I} \frac{1}{|I|^a} \| \int_{I} |f-P_{I}|^{1/p} \]

where the sup is taken over all intervals \(I\) which contain \(x\).

The maximal function \(f_{a,p}^{b}\) measures the smoothness of \(f\) and is

connected to many classical problems in analysis (see [4]). For example, for

\(k \geq (a) + 1\), we have the inequality,
(3.2)  \[ |A_n^k(f,x)| \leq c h^a \sum_{v=0}^{k} f^b_{a_i p}(x+vh), \quad h > 0, \text{ a.e. in } x. \]

The proof of (3.2) is simple and illustrative, so we should at least sketch it. If \( I \supset I^* \) are two intervals with \( |I| \leq 2|I^*| \) then

\[
(3.3) \quad \|f_p - f_{I^*}f_{I^*}^p\|_{L^p(I^*)} \leq c|I|^{-1/p}\|f_p - f_{I^*}f_{I^*}^p\|_{L^p(I^*)} + \|f_{I^*}f_{I^*}^p\|_{L^p(I^*)} \\
\leq c|I|^{-1/p} + \|f_{I^*}f_{I^*}^p\|_{L^p(I^*)} \\
\leq c|I|^a \inf_{I^*} f_{a_i p}^b.
\]

Here, the first equality is a comparison of polynomial norms (see [4, Lemma 3.1]). The inequality (3.3) holds without the restriction \( |I| \leq 2|I^*| \) since given any \( I \) and \( I^* \), we can choose \( I_0 = I \supset I_1 \supset \cdots \supset I_n = I^* \) with \( |I_j| = 2|I_{j+1}|, \quad j=0,\ldots,n-2 \) and \( |I_{n-1}| \leq 2|I_n| \). Then using (3.3), we find

\[
(3.4) \quad \|f_p - f_{I^*}f_{I^*}^p\|_{L^p(I^*)} \leq \inf_{I^*} \sum_{j=0}^{n} \|f_p - f_{I_j}f_{I_j}^p\|_{L^p(I^*)} + \|f_{I^*}f_{I^*}^p\|_{L^p(I^*)} \\
\leq c|I|^a \inf_{I^*} f_{a_i p}^b.
\]

It follows from the Lebesgue differentiation theorem (see [4, Lemma 4.1]) that

\[
(3.5) \quad \lim_{I^* \uparrow (x)} P_{I^*}f(x) = f(x) \text{ a.e.}. \]

Hence for such \( x \), taking a limit in (3.4) gives

\[
(3.6) \quad |P_{I^*}f(x) - f(x)| = \lim_{I^* \uparrow (x)} |P_{I^*}f(x) - P_{I^*}f(x)| \leq c|I|^a f_{a_i p}^b(x).
\]

This gives (3.2) since given \( x,\ldots,x+kh \), we choose \( I \) so that \( |I| = kh \) and \( x,\ldots,x+kh \in I \). Since \( \deg P_{I^*}f < k \),

\[
|A_n^k(f,x)| = |A_n^k(f_p - P_{I^*}f, x)| \leq c \sum_{v=0}^{k} f^b_{a_i q}(x+vh), \quad h > 0, \text{ a.e. in } x.
\]

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We see from (3.2) that \( f^{b}_{a,p} \) behaves like a (fractional) derivative of \( f \). In fact when \( a \) is an integer we have [4]

\[
(3.7) \quad c_{1} f^{(a)}(x) \leq f^{b}_{a,1}(x) \leq c_{2} M(f^{(a)})(x) \text{ a.e.}
\]

with \( c_{1}, c_{2} \) depending only on \( a \). Similar inequalities can be proved for other values of \( p \).

Let us define the space of functions \( C^{a}_{p} \) as the set of functions \( f \) for which

\[
|f|_{C^{a}_{p}} = \|f\|_{p} + |f|^{a}_{p} ; \quad |f|^{a}_{p} = \inf_{a,p} \|f^{b}_{a,p}\|_{p}
\]

are finite. These are smoothness spaces of order \( a \) in \( L_{p} \). For example, it follows from (3.7) that

\[
(3.8) \quad C^{k}_{p} = W^{k}_{p}, 1 < p \leq \infty; k = 1, 2, \ldots .
\]

Also, it is easy to show that \( C^{a}_{p} = L^{p} \) for all \( a > 0 \). Hence the \( C^{a}_{p} \) are generalizations of the Sobolev spaces to all \( a, p > 0 \). For any \( p \) and \( a \) non-integral we have the embeddings

\[
(3.9) \quad B^{a,p}_{p} + C^{a}_{p} + B^{a,\infty}_{p}
\]

with \( B^{a,p}_{p} \) the Besov spaces (see [4]).

We now want to show how \( f^{b}_{a,p} \) can be used to extend inequality (1.5). We can no longer use ordinary derivatives in the definition of \( P_{x} \) and therefore we begin by introducing the notion of Peano derivatives.

The proof of (3.4) together with Markov's inequality shows that for \( I \subset I^{*} \)

\[
(3.10) \quad \|D^{j}(P_{I}f - P_{I^{*}}f) I_{a_{p}}(I^{*})\|_{L_{p}(I^{*})} \leq c |I|^{-j} \inf_{I^{*}} \|f^{b}_{a,p}\|_{I^{*}}.
\]

Therefore, if \( f^{b}_{a,p}(x) = \infty \), then

\[
(3.11) \quad D^{j}f(x) = \lim_{I \uparrow \{x\}} D^{j}_{I}f(x)
\]

exists \( j = 0, \ldots, (a) \). The \( D^{j}f \) are called the Peano derivatives of \( f \). When these Peano derivatives exist, we define
The following lemma generalizes (1.5).

**Lemma 3.1.** Suppose \( 1 \leq q \leq \infty; 0 < p \leq q \) and \( \tau = \alpha + \frac{1}{q} - \frac{1}{p} > 0 \). If \( f \in \mathcal{C}_p^\alpha \), then for almost all \( x \) and all intervals \( I \) containing \( x \), we have

\[
(3.12) \quad P_x f(y) = \sum_{j=0}^{(q)} \frac{(y-x)_j}{j!} \mathcal{D} f(x).
\]

\[
(3.13) \quad \|f\|_{L_q(I)} \leq c |I|^{\alpha+1/q} M_p(f)(x)
\]

where \( F = f^b_{a,\alpha} \) and \( M_p(g) = (M(|g|^p))^{1/p} \).

**Proof.** We first estimate \( \|f\|_{L_q(I)} \). Let \( I_0 = \{I\} \) and in general the set of intervals \( I_j \) is gotten from \( I_{j-1} \) by halving the intervals in \( J_j \). Define

\[
S_j = \sum_{J_j \in I_j} P_J f(x).
\]

Then \( S_0 = f \) and \( S_j + f \) a.e. on \( I \) because of (3.5). It follows from the argument in (3.3) that whenever \( J \in I_{j-1} \) and \( J^* \subset J \) with \( J^* \subset J \), we have

\[
\|P_J f - P_{J^*} f\|_{L_q(I)} \leq c |J|^{\alpha+1/q} \inf_{J^*} F
\]

\[
\leq c |J|^{\tau} \left( \int_{J^*} f^p \right)^{1/p}.
\]

Therefore,

\[
\|S_j - S_{j-1}\|_{L_q(I)} \leq c |I|^{\tau} (2^{-J})^{1} \left( \sum_{J \in I_j} \left( \int_{J} f^p \right)^{q/p} \right)^{1/q}
\]

\[
\leq c |I|^{\tau} (2^{-J})^{1} \left( \int_{I} f^p \right)^{1/p}
\]

where in the last inequality we used the fact that an \( L_{q/p} \) norm is smaller than an \( L_1 \) norm \( q/p > 1 \). This last inequality gives
\begin{equation}
\| f - F \|_{L^q(I)} = \lim_{j \to \infty} \| S_j \cdot f \|_{L^q(I)} \leq \sum_{j=0}^{\infty} \| S_j \cdot f \|_{L^q(I)}
\leq c|I|^\alpha \int |f|^{1/p} \frac{1}{|I|^{1/p}}
\leq c|I|^\alpha \cdot M_p(F)(x).
\end{equation}

In view of (3.10) and (3.11)

\[|D^j(f)(x) - D^j(F)(x)| \leq c|I|^{\alpha - 1} F(x)\]

and hence

\begin{equation}
\| f - F \|_{L^q(I)} \leq |I|^{1/q} \| f - F \|_{L^q(I)}
\leq c|I|^{1/q} \sum_{j=0}^{\infty} |D^j(f)(x) - D^j(F)(x)| \frac{|I|^j}{j!}
\leq c|I|^{\alpha + 1} F(x) \text{ a.e. } x.
\end{equation}

Since \( F \leq M_p(F) \text{ a.e.}, (3.15) \) combines with (3.14) to give (3.3). \( \square \)

4. Further Results on Rational Approximation. We need to modify slightly the partition of unity (2.3). Again, let \( I_j, j=1, \ldots, m \) be intervals with disjoint interiors whose union is \([0,1]\) and let \( \xi_j \in I_j, j=1, \ldots, m \). We define for \( k = (a)+3 \)

\[ \phi_j(y) = (1+|I_j|^{-2}(y-\xi_j)^2)^{-k}; \phi = \sum_{j=1}^{m} \phi_j \]

and

\begin{equation}
R_j = \phi_j / \phi.
\end{equation}

Then, the \( R_j \) satisfy

\begin{enumerate}
\item \( \sum_{j=1}^{m} R_j \equiv 1 \text{ on } [0,1] \)
\item \( |R_j(y)| \leq 2^k (1+|I_j|^{-2}(y-\xi_j)^2)^{-k} \leq c(1+|I_j|^{-1}|y-\xi_j|)^{-2k} \).
\end{enumerate}
Here the last inequality uses the fact that $$(1+x^2)^{-k} \leq c(1+x)^{-2k}, \ x \geq 0.$$ 

**Theorem 4.1.** Let $a, p > 0; 1 \leq q \leq \infty$ and $p > q: = (a + \frac{1}{q})^{-1}$. If $f \in C_p^*$ then

$$r_n(f)_q \leq c|f|_p \alpha^{-a}$$

with $c$ independent of $n$ and $f$.

**Proof.** The maximal function $M$ boundedly maps $L_p$ into $L_r$ for all $r < p$. Choose $a < r < p$. Then, there is a constant $c_0$ such that

$$\|M_p(f)\|_r \leq c_0|f|_p$$

with $p: = \frac{2}{a+1}$. It will be enough to prove (4.3) for functions $f$ which satisfy $|f|_a = c_0^{-1}$. In this case, $M_p(f) \leq 1$ and hence we can choose intervals $I_1, \ldots, I_{2n}$ such that

(4.4)  
1) the $I_j$ have disjoint interiors and $[0,1] = \bigcup_{j=1}^{2n} I_j$  
2) $|I_j| \leq n^{-1}$, $j=1, \ldots, 2n$  
3) $\int_{I_j} [M_p(f)]^r \leq n^{-1}$.

For each $j=1, \ldots, 2n$, choose a point $\xi_j \in I_j$ such that

$$M_p(f)(\xi_j) = \inf_{I_j} M_p(f).$$

Then,

(4.5)  
$$M_p(f)(\xi_j) \leq \frac{1}{|I_j|} \int_{I_j} M_p(f)^{1/r} \leq (n|I_j|)^{-1/r}.$$

Let $P_{\xi_j}$ be the polynomial in Lemma 3.1 for $x = \xi_j$. The rational function

$$R: = \sum_{j=1}^{2n} \frac{P_{\xi_j}(f)}{I_j}$$

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has degree \( \leq 8kn \) and it will be enough to show that

\[
(4.6) \quad \text{if-RI}_{q} \leq cn^{-\alpha}
\]

with \( c \) independent of \( f \) and \( n \).

The proof of (4.6) is similar to the proof of Theorem 2.1. In fact, the case \( q = \infty \) is identical and so we proceed only with the case \( q < \infty \). Let \( I_{v} \) be the set of those intervals \( I_{j} \) such that \( 2^{-\nu+1}n^{-1} < |I_{j}| \leq 2^{-\nu}n^{-1} \). Then, for any interval \( I_{1} \), we have

\[
(4.7) \quad \text{if-RI}_{L_{1}}(I_{1}) \leq \sum_{v} \text{S}_{v}^{L_{1}}(I_{1})
\]

where \( S_{v} = \sum_{I_{j} \in I_{v}} (f-P_{\xi_{v}})R_{j} \).

Suppose \( I_{1} \subset I_{v} \). Fix \( 1 \) for the moment. We want to estimate \( \text{S}_{v}^{L_{1}}(I_{1}) \). It will be convenient to write \( S_{v} = S_{v}^{-} + S_{v}^{+} \) where \( S_{v}^{-} \) is the sum over all intervals in \( I_{v} \) to the left of \( I_{1} \) and \( S_{v}^{+} \) is the sum over all intervals in \( I_{v} \) to the right of \( I_{1} \). We now estimate \( \text{S}_{v}^{L_{1}}(I_{1}) \); the estimate for \( S_{v}^{+} \) is the same. If \( \nu > v \), we let \( J_{0} = I_{1} \). If \( \nu = v \), we take \( \nu+1 = 2^{\nu}n \) and let \( J_{0}, \ldots, J_{m} \) be disjoint intervals of length \( (\nu+1)^{-1}|I_{1}| \) whose union is \( I_{1} \) and which are ordered from left to right. It follows that for \( I_{j} \in I_{v} \),

\[
d_{ij} + s|I_{j}| \leq |y-x_{j}| \leq 2|I_{j}| + d_{ij} + s|I_{j}| \text{ when } y \in J_{g}
\]

where \( d_{ij} = \min\{|a-b|: a \in I_{i}, b \in I_{j}\} \). If \( J_{g} \) is the smallest interval which contains \( J_{g} \) and \( x_{g} \), then \( |J_{g}| \leq d_{ij} + (s+2)|I_{j}| \). Using these facts with (3.13), (4.2) (ii) and (4.5), we have, with

\[
a_{n,v} = \frac{2^{-\nu}n^{1-1/\tau}}{(\nu+1)^{-1} - (\alpha+1)/q},
\]

\[
(4.8) \quad \text{S}_{v}^{L_{1}}(J_{g}) \leq \sum_{I_{j} \in I_{v}} (f-P_{\xi_{v}})R_{j}^{L_{1}}(J_{g})
\]

\[
\leq \sum_{I_{j} \in I_{v}} (1+s+d_{ij}|I_{j}|^{-1})^{-2k+\alpha+1/q}
\]

\[
\leq \sum_{I_{j} \in I_{v}} (1+s+1)^{-2k+\alpha+1/q} \leq \sum_{I_{j} \in I_{v}} (1+s)^{-2k+\alpha+1/q} \leq \sum_{I_{j} \in I_{v}} \frac{1}{(1+s)^{1-q}} \leq \frac{1}{(1+s)^{1-q+1}}.
\]
Since \( r_1 = 2k - a - \frac{1}{q} - 1 > 1 \), we can use (4.8) and a similar estimate for \( S^a \) to find

\[
(4.9) \quad IS^a_{L_q(L_1)} \leq cn_{n,\nu} \left( \sum_{a=1}^{\infty} (1+\alpha)^{-\alpha q} \right)^{1/q} = cn_{n,\nu}.
\]

Using this back in (4.7) shows that \( \lim_{q \to \infty} \frac{\mu-I_{L_q(L_1)}}{q} \leq cn_{n,\nu}^{-(\alpha+1/q)} \), and therefore

\[
\lim_{q \to \infty} \frac{\mu-I_{L_q(L_1)}}{q} \leq cn_{n,\nu}^{-(\alpha+1/q)} \left( \sum_{a=1}^{\infty} \frac{1}{a} \right)^{1/q} \leq cn_{n,\nu}^{-\alpha}. \quad \square
\]

**Corollary 4.2.** If \( 1 \leq p, q \leq \infty \) and \( k \) is a non-negative integer, then for each \( f \in W^k_p \), we have

\[
r_n(f) \leq c_{k} \frac{1}{n^k}.
\]

**Proof.** If \( p > 1 \), this follows from Theorem 4.1 and (3.8). When \( p = 1 \), the same proof as Theorem 2.1 gives the result. \( \square \)

**Corollary 4.3.** If \( \alpha, p > 0; p \leq q \leq \infty \), and \( \alpha + \frac{1}{p} - \frac{1}{q} > 0 \), then whenever \( \alpha \) is not an integer and \( f \in B^\alpha_p \), we have

\[
r_n(f) \leq c_{\alpha, p, n} \frac{n^{-\alpha}}{n}.
\]

**Proof.** This follows from Theorem 4.1 and (3.9). \( \square \)

**References**


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