The $n$-Width of $BV \cap \text{Lip } \alpha$

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We determine asymptotically the $n$-width of the space of functions $BV \cap \text{Lip } \alpha$ as a subset of $C$. Since the $n$-width behaves like $n^{-(t-1)/2}$, it is seen that this class of functions has significantly smaller $n$-width than $\text{Lip } \alpha$.

There are many theorems of analysis which show that the space $S_\alpha := BV \cap \text{Lip } \alpha$ is small when compared to $\text{Lip } \alpha$. For example, we know that each $f \in S_\alpha$ has an absolutely convergent Fourier series if $\alpha > 0$ but this is not the case for $\text{Lip } \alpha$ unless $\alpha = 1$. In this note, we shall measure the size of $S_\alpha$ by computing the $n$-width of the unit ball of $S_\alpha$ in the space $C_{2\pi}$ of $2\pi$ periodic continuous functions.

One way to measure the size of a compact set $K$ of a Banach space $X$ is through the notion of $n$-widths. If $X_n$ is an $n$-dimensional subspace of $X$ then

$$E(f, X_n) := \inf_{x_n \in X_n} \| f - x_n \|_X$$

is the error in approximating $f$ by elements of $X_n$ and

$$E(K, X_n) := \sup_{f \in K} E(f, X_n)$$

is the error in approximating the compact set $K$. The $n$-width of $K$ looks for best $n$-dimensional subspaces:

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\( d_n(K)_{\mathcal{X}} := \inf_{\mathcal{X}_n} E(K,\mathcal{X}_n) \).

The asymptotic behavior of \( d_n(K)_{\mathcal{X}} \) gives some idea of the size of \( K \) in \( \mathcal{X} \). For example if \( C_{2\pi} \) denotes the space of continuous \( 2\pi \) periodic functions, then the unit ball \( U(\text{Lip } \alpha) \) of \( \text{Lip } \alpha \) has \( n \)-width

\[
(2) \quad d_n(U(\text{Lip } \alpha))_{C_{2\pi}} \approx n^{-\alpha}, \quad n \to \infty.
\]

Actually, there is a much larger set than \( U(\text{Lip } \alpha) \) which also has \( n \)-width \( n^{-\alpha} \). It was shown by B. S. Kashin [1] that the Sobolev space \( W^2_\alpha \) satisfies:

\[
(3) \quad d_n(U(W^2_\alpha))_{C_{2\pi}} \approx n^{-\alpha}, \quad \alpha > \frac{1}{2}.
\]

Since \( W^2_\alpha = B^{2\alpha, 2}_2 \) (the Besov space), it follows from an interpolation of linear operators argument that

\[
(4) \quad d_n(U(B^{2\alpha, r}_2)) \approx n^{-\alpha}, \quad \alpha > \frac{1}{2}; \quad r > 0.
\]

Since \( \text{Lip } \alpha \subset B^{2\alpha, r}_2 \), (3) is a (substantial) improvement over (2).

The main result of this note is to determine the \( n \)-width of the set \( K_\alpha := \{ f \in S_\alpha : \max \{ \| f \|_{BV}, \| f \|_{\text{Lip } \alpha} \} \leq 1 \} \) in \( C_{2\pi} \). The following result should be compared with (2).

**Theorem 1.** For each \( 0 < \alpha < 1 \), there are constants \( C_1, C_2 > 0 \) such that

\[
(5) \quad C_1 n^{-(\alpha+1)/2} \leq d_n(K_\alpha)_{C_{2\pi}} \leq C_2 n^{-(\alpha+1)/2}.
\]

**Proof.** The upper estimate follows from an embedding of \( K_\alpha \) into a Besov space. If \( f \in K_\alpha \) and \( h > 0 \), we have

\[
(6) \quad \int_0^{2\pi} |f(x+h) - f(x)|^2 dx \leq h^\alpha \int_0^{2\pi} |f(x+h) - f(x)| dx \leq h^{\alpha+1}.
\]
The last inequality is apparent for absolutely continuous functions and follows for any \( f \) in \( K_{\alpha} \) since such an \( f \) is the uniform limit of absolutely continuous functions from \( K_{\alpha} \).

It follows from (5) that \( K_{\alpha} \subseteq U(B_{2}^{B_{2}^{\infty}}) \), \( \beta = (\alpha+1)/2 \) and so from (3),

\[
d_{n}(K_{\alpha}) \leq d_{n}(U(B_{2}^{B_{2}^{\infty}})) \leq C n^{-\beta} = n^{-(\alpha+1)/2}
\]

which is the right hand side of (4).

To prove the lower estimate in (4), we shall use an idea of W. Rudin [2] and construct sufficiently many functions in \( S_{\alpha} \) which are mutually orthogonal. In part, the orthogonality will come from making supports disjoint and in part, by using Latin squares to guarantee orthogonality where supports are not disjoint.

It is enough to prove (4) when \( n \) is a 2 power. Let \( N = 2n \) and choose \( m \) as the largest 2 power such that

\[
2m \leq (2N/n)^{\alpha}
\]

It follows that

\[
m \geq \frac{1}{2}(2N/n)^{\alpha}
\]

Also \( m \leq N \).

Now divide \([0,2\pi]\) into \( N/m \) intervals \( I_{\nu} = [x_{\nu}, x_{\nu+1}] \) \( \nu = 0,1,\ldots, (N/m)-1 \): \( x_{\nu} = 2\pi m/N. \) For each \( \nu \) we choose points \( x_{\nu} = x_{\nu} + (2\pi-1)\pi/N, \mu = 1,\ldots, m \) from \( I_{\nu} \).

We will also need a Latin square

\[
A = (a_{ij}) \quad a_{ij} = \pm 1
\]

of dimension \( m \times m \). Let \( v_{1}, \ldots, v_{m} \) denote the columns of \( A \). Then, these vectors are orthogonal.

Here is how we construct our orthogonal functions. If \( 1 \leq i \leq N \), we write
Define $g_i$ as the piecewise linear function

$$g_i := \sum_{\mu=1}^{m} a_{\mu k} \psi_{\mu k}$$

with $\psi_{\mu k}$ the "hat function" which is supported on the interval $[x_{\mu k} - \pi/N, x_{\mu k} + \pi/N]$ has value zero at the end points of this interval and 1 at the point $x_{\mu k}$.

It is easy to see that the $g_i$ are orthogonal. Indeed, suppose that $i_1 = k_1 m + l_1$ and $i_2 = k_2 m + l_2$. If $k_1 \neq k_2$ then the $g_i$ are orthogonal because they have disjoint supports. If $k_1 = k_2$, then

$$\int_0^{2\pi} g_{i_1} g_{i_2} = \frac{\pi}{6N} v_{l_1} \cdot v_{l_2} = 0$$

because the columns of $A$ are orthogonal.

The functions $g_i$ have $L_2$ norm $(2\pi m)_i^\frac{1}{2}$. Therefore, the normalized functions

$$f_i := (\frac{3N}{2\pi m})^\frac{1}{2} g_i$$

form an orthonormal system.

Now suppose $X_n$ is any $n$-dimensional subspace of $C_{2\pi}$ and let $\phi_1, \ldots, \phi_n$ be an orthonormal basis for $X_n$. We extend this orthonormal system by adjoining functions $\phi_{n+1}, \ldots$ to arrive at a complete orthonormal system. Now consider the matrix $B$ with

$$b_{ij} := (f_i, \phi_j)^2$$

The sum of the entries in a given row of $B$ is one and the sum of the entries in a given column is at most one. Hence

$$\sum_{i=1}^{N} E(f_i, X_n)^2 = \sum_{i=1}^{N} \sum_{j=n+1}^{\infty} (f_i, \phi_j)^2 = N - \sum_{i=1}^{N} \sum_{j=1}^{n} (f_i, \phi_j)^2$$
\[ \geq N - n = n. \]

As a result

(7) \[ \max_{1 \leq i \leq N} E(f_i, X_n)^2 \geq \frac{1}{2}. \]

It is easy to check that \( \| g_i \|_{BV} = 2m \) and \( \| g_i \|_{\text{Lip } \alpha} = \left( \frac{N}{\pi} \right)^{\alpha} \).

Hence the function \( \tilde{f_i} = (\frac{m}{N})^\alpha g_i = \left( \frac{m}{N} \right)^{\frac{\alpha}{2} + \frac{1}{2}} f_i \) satisfy

(8) \[ \| \tilde{f_i} \|_{BV} \leq \| \tilde{f_i} \|_{\text{Lip } \alpha} = 1 \]

so that \( \tilde{f_i} \in K_{\alpha} \).

According to (7) and (8)

\[ \sqrt{2\pi} E(K_\alpha, X_n)_{C_{2\pi}} \geq E(K_\alpha, X_n)_{L^2} \geq \max_{1 \leq i \leq n} E(f_i, X_n)_{L^2} \geq \frac{1}{\sqrt{2}} \left( \frac{m}{N} \right)^{\frac{\alpha}{2} + \frac{1}{2}} \]

\[ \geq C N^{-\frac{\alpha}{2} - \frac{1}{q}} \geq C N^{-\frac{(\alpha+1)}{2}} \geq C n^{-(\alpha+1)/2}. \]

Since \( X_n \) is arbitrary this gives the lower estimate in (4).

The approach above extends readily to the more general compact set

(9) \[ K_{\alpha, \beta} := \{ f : \| f \|_{B^\alpha p, r} \leq 1 ; \| f \|_{B^\beta q, s} \leq 1 \} ; \frac{1}{p} + \frac{1}{q} = 1 ; p \leq q \]

with \( \alpha, \beta, r, s > 0 \) and \( \| \cdot \|_{B^\alpha p, r} \) the usual Besov space semi-norms.

The space \( B^\alpha p_{\text{loc}} \subset B^\beta q_{\text{loc}} \) if \( \alpha > \beta + 1/p - 1/q \) and \( B^\beta q_{\text{loc}} \subset B^\alpha p_{\text{loc}} \) if \( \beta > \alpha \).

Thus, the only interesting cases are when

\[ \beta < \alpha \leq \beta + 1/p - 1/q. \]
Theorem 2. If $K_{\alpha,\beta}$ is defined as in (9) with $1 \leq p \leq 2$, and $\beta < \alpha < \beta + 1/p - 1/q$, then there are constants $C_1, C_2 > 0$ such that

$$C_1 n^{-(\alpha+\beta)/2} \leq d_n(K_{\alpha,\beta})_{C_{2\pi}} \leq C_2 n^{-(\alpha+\beta)/2}$$

Proof. The proof is essentially the same as Theorem 1 and so we only indicate the necessary changes. Let $\Delta_h^k$ denote the $k$-th difference with step size $h$. If $f \in K_{\alpha,\beta}$ and $k = [\alpha+\beta] + 1$, then

$$\int_0^{2\pi} |\Delta_h^k(f,x)|^2 \, dx \leq \left( \int_0^{2\pi} |\Delta_h^k(f,x)|^p \, dx \right)^{1/p} \left( \int_0^{2\pi} |\Delta_h^k(f,x)|^q \, dx \right)^{1/q} \leq C_0 h^{\alpha} ||f||_{B^{\alpha+\beta}_{p,q}} h^{\beta} ||f||_{B^{\beta}_{p,q}} \leq C_0 h^{\alpha+\beta}$$

with $C_0$ depending only on $k$. Hence $K_{\alpha,\beta}$ is contained in a ball of radius $\sqrt{C_0}$ in $B^{\alpha+\beta}_{p,q}$ with $C_0$ depending only on $k$. It then follows from (3) that

$$d_n(K_{\alpha,\beta})_{C_{2\pi}} \leq \sqrt{C_0} \, d_n(U(B^{\alpha+\beta}_{p,q}))_{C_{2\pi}} \leq C n^{-(\alpha+\beta)/2}$$

which is the upper estimate in (10).

For the lower estimate, we argue similarly to Theorem 1 except that we need smoother functions $f_i$. Let $\phi$ be a non-negative function in $C^\infty$ which is supported on $[-1,1]$ with

$$\int_{-\infty}^{\infty} \phi^2 = 1; ||\phi||_{\infty} \leq 2$$

We suppose as before that $n$ is a 2 power, $N = 2n$ and we define $m$ as the largest 2 power such that $m \leq N^{\gamma+1}$ with $\gamma = (\beta-\alpha)/(1/p-1/q)$. Note that $-1 \leq \gamma < 0$. The points $x_{\mu\nu}$ are defined as in Theorem 1. Finally we define $h_0 = 2\pi/N$.

The function $\phi_{\mu\nu} = m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} \phi \left( \frac{x-x_{\mu\nu}}{h_0} \right)$ is supported on $[x_{\mu\nu} - \frac{1}{4} h_0, x_{\mu\nu} + \frac{1}{4} h_0]$. If $1 \leq i \leq N$, we write $i = km + \lambda$ with
$1 \leq \ell \leq m$ and define

$$f_i = \sum_{\mu=1}^{m} a_{\mu \ell} \phi_{\mu k}$$

where $a_{\mu \nu}$ are the entries of the Latin square $A$.

It follows that the $f_i$ are an orthogonal system and therefore as in (7), for any $n$-dimensional space $\mathcal{X}_n$.

$$\max_{1 \leq i \leq n} E(f_i, x_n)^2_{C_{2\pi}} \geq \frac{1}{4}.$$ (11)

We now check the norm of $f_j$ in $E_{p, k}^\alpha$. Let $k > \alpha$. Now, $\|\phi_{\mu \nu}(k)\|_\infty \leq C m^{-\frac{1}{2}} h_0^{-k-\frac{1}{2}}$ and since the $\phi_{\mu \nu}$ have disjoint supports and $|a_{\mu \nu}| = 1$,

$$|\Delta_h^k(f_j, x)| \leq C \max (m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^{k}, 1) \text{ on support of } f_i.$$ (12)

Hence,

$$\|\Delta_h^k(f_j)\|_p \leq C \max (m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^{k}, 1) (m/N)^{1/p}$$

It follows that the modulus of smoothness $\omega_k(f_j)_p$ satisfies

$$\omega_k(f_j, h)_p \leq C (m/N)^{1/p} \begin{cases} m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (h/h_0)^{k}, & h \leq h_0, \\ 1, & h \geq h_0. \end{cases}$$

Hence,

$$|f_j|_{\alpha, r} \leq C \left( h^{-\alpha} \omega_k(f_j, h)_p \right)^{r} \left( \frac{dh}{h} \right)^{1/r} \leq C (m/N)^{1/p} \left[ m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} h^{-\alpha} + h^{-\alpha} \right]$$

$$\leq C (m/N)^{1/p} m^{-\frac{1}{2}} h_0^{-\frac{1}{2}} h^{-\alpha}$$

$$\leq C N^{(1/p-1)} N^\alpha \leq C N^{(\alpha+\beta)/2}$$

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The same computation shows that $|f_i|_{p} < C n^{(\alpha+\beta)/2}$.

The functions $g_i = C^{-1} n^{-(\alpha+\beta)/2} f_i$ are in $K_{\alpha,\beta}$ and according to (11) give the lower estimate in (10). \(\square\)

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REFERENCES
