THE $K$ FUNCTIONAL FOR $(H_1, BMO)$

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1. Introduction. There are several theorems [1], [4], [5] which show that in some sense $H_1$ and $BMO$ can serve as replacements for $L_1$ and $L_\infty$ respectively for interpolation theory. For example, it is known that the $L_p$ spaces $1 < p < \infty$ are interpolation spaces between any of the pairs $(X_1, X_\infty)$ with $X_1$ either $L_1$ or $H_1$ and $X_\infty$ either $L_\infty$ or $BMO$. We are interested in the finer question of characterizing the $K$ functionals for these pairs $(X_1, X_\infty)$. Actually the $K$ functional is known or easily derived from known results in all but the one case $(H_1, BMO)$. The characterization of the $K$ functional for this latter pair is the main result of this paper.

Recall that for any pair of Banach spaces $(X, Y)$, the Peetre $K$ functional is defined for $f \in X + Y$ by

$$K(f, t, X, Y) := \inf_{h+g} (||h||_X + t||g||_Y), \quad t > 0.$$ 

Perhaps, it is useful to explain the interest in characterizing these $K$ functionals. If $T$ is a bounded operator on $X_1$ and $X$ then $T$ satisfies

$$K(Tf, t) \leq cK(f, t) \quad \text{for all } f, \quad X_1 + X_\infty$$

(1.1)

with $K(f, t) := K(f, t, X_1, X_\infty)$ the corresponding $K$-functional. The inequality (1.1) carries more information than any particular result on mapping of spaces. For example, $K(f, t, L_1, L_\infty) = tf^{**}(t)$ with $f^*$ the decreasing rearrangement of $f$ and $f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)ds$; hence if $T$ is bounded on $L_1$ and $L_\infty$, then

$$K(Tf, t) \leq cK(f, t).$$

(1.2)

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2 All spaces are over $\mathbb{R}^n$ unless specifically stated otherwise.
It follows from (1.2) that \( T \) is bounded (for example) on \( L \log L \); a result which is not included in the usual interpolation theorems for \((L^1, L^\infty)\) which give only that \( T \) is bounded on \( L^p, 1 < p < \infty \).

Another reason for studying \( K \) functionals is that they usually involve analytic quantities which are fundamental to the study of the particular pairs of spaces; \( f^{**} \) for \((L^1, L^\infty)\). Another example is \((L^1, \text{BMO})\) where C. Bennett and R. Sharpley [1] have shown

\[
(1.3) \quad K(f, t, L^1, \text{BMO}) \approx tf^{**}(t) \quad \text{for all} \quad f \in L^1 + \text{BMO} \quad \text{and} \quad t > 0
\]

with

\[
f^{*}(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q|; \quad f_Q := \frac{1}{Q} \int_Q f
\]

the Fefferman-Stein sharp function. We use the notation "\( \approx \)" to indicate that the quotient of the two expressions are bounded away from 0 and \( \infty \) (independent of \( f \) and \( t \) in (1.3)). The fact that \( L^p \) spaces \( 1 < p < \infty \) are interpolation spaces between \( L^1 \) and \( \text{BMO} \) follows from (1.3) and the fact that for \( 1 < p < \infty \)

\[
(1.4) \quad \|f^*\|_{L^p} \approx \|f\|_{L^p} \quad \text{for all} \quad f \in L^p.
\]

It is possible to characterize the \( K \) functional for \((H^1, L^\infty)\) from the work of C. Fefferman-N. Rivière and Y. Sagher [4]. They have shown that each (smooth) function \( f \) can be written as \( f = b + g \) with

\[
\|b\|_{H^1} + t \|g\|_{L^\infty} \leq c \int_0^t (Mf)^*(s) \, ds - ct(Mf)^**(t)
\]

with \( M \) the grand maximal function (see (2.11)). From this it follows that

\[
K(f, t, H^1, L^\infty) \leq ct(Mf)^**(t).
\]

On the other hand, \( F + (MF)^** \) is subadditive and \( M \) maps \( H^1 \) into \( L^1 \) and \( L^\infty \) into \( L^\infty \); hence for \( f = b + g \)

\[
t[Mf]^**(t) \leq t(Mb)^**(t) + t(Mg)^**(t) \leq c[Mb]_{L^1} + t[Mg]_{L^\infty}
\]

\[
\leq c(\|b\|_{H^1} + t \|g\|_{L^\infty}).
\]
Taking an inf over all such decompositions gives
\[ t(Mf)_{**}(t) \leq cK(f,t,H_1,L_\infty) \]
or
\[ K(f,t,H_1,L_\infty) \sim t(Mf)_{**}(t), \quad f \in H_1 + L_\infty. \tag{1.5} \]

We shall characterize the \( K \) functional for the remaining pair \((H_1, \text{BMO})\). This characterization involves a generalization of the sharp function \( \hat{f}^{\#} \) to a new sharp function \( \hat{f}^{\#}_{H_1} \) which essentially replaces the role of \( L_1 \) by that of \( H_1 \) (see §2 for a precise definition). We then show that
\[ K(f,t,H_1,\text{BMO}) \sim t\hat{f}^{\#}_{H_1}(t) \tag{1.6} \]

As a consequence, any operator \( T \) which is bounded on \( H_1 \) and \( \text{BMO} \) satisfies
\[ (Tf)_{H_1}^{\#*} \leq c \hat{f}^{\#*}_{H_1}. \tag{1.7} \]

In the process of proving (1.6), we establish several results which compare \( \hat{f}^{\#}_{H_1} \) with other sharp functions. These in turn show that \( \| \hat{f}^{\#}_{H_1} \|_{L_p} \sim \| f \|_{L_p} \) (Corollary 3.4). Therefore the fact that \( L_p, 1 < p < \infty \) is an interpolation space between \( H_1 \) and \( \text{BMO} \) follows from (1.7) by applying \( L_p \) norms.

2. Sharp Functions. \( \text{BMO} \) is the space of all \( f \) satisfying
\[ \| f \|_{\text{BMO}} := \| \hat{f}^{\#} \|_{L_\infty} < \infty. \tag{2.1} \]

It is a very useful fact that \( \hat{f}^{\#} \) can be replaced in (2.1) by (see [1, Cor. 4.7])
\[ \hat{f}^{\#}(x) := \sup_Q \left( \frac{1}{|Q|} \int_Q |f-f_Q|^p \right)^{1/p}. \tag{2.2} \]

A word of explanation about notation; we are being consistent with [2] where maximal functions \( f^{\#}_{\alpha,p} \), \( \alpha, p > 0 \) are introduced.

There is another important variant in the definition of \( \hat{f}^{\#} \). If we let \( \mathcal{P}_N \)
denote the space of polynomials of total degree $N$, then for any fixed $N$ we have 
\[ (2.3) \quad \|f\|_{0,p}^p \preceq T; \quad F(x) := \sup_{Q \ni x} \inf_{\phi \in \mathcal{P}_N} \frac{1}{|Q|} \int_Q |f - \phi|^p. \]

We want now to introduce a sharp function which replaces the role of $L_p$ in the definition of $f_{0,p}^p$ by the space $H_1^\perp$. We must take some care since $(f - f_Q^\perp) \chi_Q$ is generally not in $H_1^\perp$. Let $A := 20 \sqrt{n}$ (the precise value of $A$ is not important; any $A$ sufficiently large would do). For each cube $Q$ with diameter $d$ let $\phi_Q$ denote the set of functions $\phi$ satisfying
\[ (2.4) \quad \begin{array}{l}
1) \quad \phi \text{ supported on } A Q; \quad 0 \leq \phi \leq 1. \\
2) \quad \phi \geq A^{-1} \quad \text{ on } A^{-1} Q. \\
3) \quad \|D^n \phi\|_{L^\infty} \leq A d^{-|n|}, \quad \forall \quad n \geq 0.
\end{array} \]

We are using the notation $\lambda Q_A$ to denote the cube with the same center as $Q$ and diameter $\lambda d$ with $d$ the diameter of $Q$.

Given $\phi \in \phi_Q$, consider the inner product
\[ (2.5) \quad (f, g, \phi) := \frac{1}{|Q|} \int_Q f g \phi, \]
and denote by $P_{\phi}$ the orthogonal projection operator from $L_1(AQ)$ onto $\mathcal{P}_N$ with $N := n + 1$, i.e. $P_{\phi} f$ is the unique polynomial in $\mathcal{P}_N$ which satisfies
\[ (2.6) \quad (f - P_{\phi} f, \tau) = 0 \quad \text{for all } \tau \in \mathcal{P}_N. \]

If $f \in H_1^\perp$, define
\[ (2.7) \quad f_{H_1^\perp} (x) := \sup_{\phi \in \phi_Q} \sup_{Q \ni x} \| (f - P_{\phi} f) \|_{H_1^\perp}. \]

As we shall see in this and the next section, $f_{H_1^\perp}$ is similar to $f_{0,p}^p$. To begin with, we recall two equivalent norms for $H_1^\perp$ from the C. Fefferman-E. Stein [3] theory. If $k \in L_1$, let
\[ (2.8) \quad \|k\|_{W_N} := \sum_{|n| \leq N} \int_{|x| \leq n} |(1 + |x|)^N |D^n k(x)| \, dx \]
with $N := n + 1$. We fix a kernel $K$ with the properties:
(2.9)  
1) \( K \geq 0 \)

ii) \( \int K = 1 \)

iii) \( K \) supported on \( |x| \leq 1 \)

iv) \( \| D^\nu K \|_{\infty} \leq c, \) for all \( |\nu| \leq N. \)

Then \( \| K \|_{N} \leq c. \) Let

\[ f^+(x) := \sup_{\epsilon > 0} |f * K_\epsilon(x)| \]

Then,

(2.10) \[ \| f^+ \|_{L^1} \approx \| f \|_{N} \]

There is another important equivalent norm for \( N \) given by the grand maximal function. Let \( \alpha > 0 \) be a fixed constant and

(2.11) \[ Mf(x) := \sup_{\|k\|_{N} \leq 1} \sup_{|x_1-x|<\alpha \epsilon} |f * k_\epsilon(x)|. \]

Then

(2.12) \[ \| Mf \|_{L_1} \approx \| f \|_{N}. \]

Note in the definition (2.11) the kernels \( k \) are not required to have integral one.

We would like now to give estimates for \( P_\phi f \) in terms of \( Mf. \) These are similar to those given in [4]. Let \( \{ \eta_i \}_{i=1}^m, n = \text{dim}(P_N) \) be an orthonormal basis for \( P_N \) with respect to the inner product \(( , )_{\phi}. \) Then

\[ P_\phi f = \sum_{i=1}^m (f, \eta_i)_{\phi} \eta_i. \]

Using (2.4)ii, we have

\[ \| \pi_1 \|_{L_2(A^{-1}Q)} \leq (A \int \pi_j^2 \phi)^{1/2} = (A|\phi|)^{1/2}. \]

It follows that for any \( \lambda > 0 \) (see [2, §3])
(2.13) \[ \| \pi_\lambda \|_{L_\infty(\lambda Q)} \leq c \| \pi_\lambda \|_{L_\infty(A^{-1}Q)} \leq c |Q|^{-\lambda} \| \pi_\lambda \|_{L_2(A^{-1}Q)} \leq c \]

with \( c \) depending only on \( \lambda, A \) and \( n \).

**Lemma 2.1.** For any \( \lambda > 0 \), there is a constant \( c > 0 \) depending at most on \( \lambda, A \) and \( n \) such that

\[ |(\xi, \pi_\lambda \phi)_{\lambda Q} | \leq c M \xi(x), \quad \text{for all } x \in \lambda Q \]

(2.15) \[ \| P_\phi \pi \|_{L_\infty(\lambda Q)} \leq c M \xi(x), \quad \text{for all } x \in \lambda Q. \]

**Proof.** Clearly (2.15) follows from (2.14) and (2.13). To prove (2.14), we notice that (2.13) and Markov's inequality give that \( \| D^\nu \pi_\lambda \|_{L_\infty(\lambda Q)} \leq c d^{-|\nu|} \| \pi_\lambda \|_{L_\infty(\lambda Q)} \), which is \( |\nu| \leq N \). Hence, using (2.4) we see that the kernel \( k(u) := \pi_\lambda (x-du) \phi(x-du) \) satisfies

\[ \| D^\nu k \|_{L_\infty} \leq c \| \pi_\lambda \|_{L_\infty} \]

If \( x \in \lambda Q \), then \( k \) is supported in \( |u| \leq A+\lambda \) and so \( \| k \|_{L_\infty} \leq c \). Therefore,

\[ |(\xi, \pi_\lambda \phi)_{\lambda Q} | = |(\xi, \pi \phi)_{\lambda Q} | = |\xi \ast k_\xi(x) | \leq c M \xi(x). \]

Our next result estimates \( \psi^+ \) when \( \psi := (\xi - P_\phi \xi) \phi \).

**Lemma 2.2.** If \( \lambda \geq 2A \), there is a constant \( c > 0 \) depending only on \( \lambda \) and \( n \) such that for each cube \( Q \) with diameter \( d \) and center \( z \) and each \( \phi \in \Phi_Q \), we have

(2.16) \[ 1) \psi^+(x) \leq c M \xi(x), \quad x \in \lambda Q \]

\[ 2) \psi^+(x) \leq c |Q|^{-\lambda} d^n A^{-1} |x-z|^{-2n-1} \inf_{\lambda Q} \| f \|_{\lambda Q}, \quad x \notin \lambda Q. \]

**Proof.** For 1), we consider two cases.

- **Case 1.** \( \rho \leq d \). In this case, \( k(u) := X(u) \phi(x-\rho u) \) satisfies

\[ \| k \|_{L_\infty} \leq c \] because of (2.4), and so

\[ |(f \phi) \ast K_\rho(x)| = |\xi \ast k_\rho(x)| \leq c M \xi(x). \]
Also from \((2.15)\) and \((2.4)\),

\[
| (\phi P_{\phi} \ast K_{e}(x) | \leq \| \phi (P_{\phi} \ast \xi) \| _{L_{\infty}} \leq \| P_{\phi} \xi \| _{L_{\infty}} (AQ) \leq c M_{f}(x) .
\]

Hence,

\[ (2.17) \quad | \psi \ast K_{e}(x) \| \leq c M_{f}(x) \]

in this case.

**Case 2.** \(e > d\). In this case, the kernel \(k(u) := K_{e}(\frac{du}{c}) \phi (x - du)\) satisfies \(\| k \| _{U_{N}^{\infty}} \leq c\) and so

\[
| (f \phi) \ast K_{e}(x) | = \left( \frac{d_{1} d_{2}}{c} \right) | f \ast K_{e}(x) | \leq c M_{f}(x)
\]

Also, from \((2.15)\) and \((2.4)\),

\[
| (\phi P_{\phi} \ast K_{e}(x) | \leq \| \phi P_{\phi} \xi \| _{L_{\infty}} \leq \| P_{\phi} \xi \| _{L_{\infty}} (AQ) \leq c M_{f}(x) .
\]

Hence \((2.17)\) holds in this case as well. Taking a sup over all \(e > 0\) in \((2.17)\) gives \(\| \|\).

To prove \(ii)\), fix \(x \notin \lambda Q\) and define \(\delta := \text{dist}(x, AQ)\). Then \(\delta \geq c|x - z|\). If \(e < \delta\), then \(\psi \ast K_{e}(x) = 0\); hence we may assume \(e \geq \delta\).

Now, there is a Taylor polynomial \(T\) of degree at most \(N\) such that

\[ (2.18) \quad | K_{e}(u - x) - T(u) | \leq c e^{-2n-1} d^{n+1} u \in \lambda Q \]

because derivatives of order \(N\) of \(K_{e}(\ast - x)\) are less than \(c e^{-2n-1}\). Thus, using \((2.18)\) and \((2.6)\) gives

\[ (2.19) \quad | \psi \ast K_{e}(x) | \leq \int (f(u) - P_{\phi} f(u)) \phi(u) K_{e}(u - x) du \]

\[
= \int (f(u) - P_{\phi} f(u)) \phi(u) [K_{e}(u - x) - T(u)] du \]

\[
\leq c \left( \frac{e^{n+1}}{e^{2n+1}} \right) \int_{AQ} \left| f - P_{\phi} f \right| du
\]

For any \(\pi \in \mathcal{P}_{N}\), \(P_{\phi}(\pi) = \pi\) and so
\[
\int_{\Lambda} |f - P_\phi f| \leq \int_{\Lambda} |f - \pi| + \int_{\Lambda} |P_\phi (f - \pi)| \\
\leq c \int_{\lambda \Lambda} |f - \pi| \leq c \int_{\lambda \Lambda} |f - \pi|
\]
because \( P_\phi \) is a bounded operator on \( L_1(\Lambda) \) (see (2.13)). Taking an inf over all \( \lambda \) and using (2.3) gives
\[
\int_{\Lambda} |f - P_\phi f| \leq c |\lambda \Lambda| \inf_{\lambda \Lambda} f^*.
\]
Using this in (2.19) completes the proof of (ii). \( \square \)

3. Lower estimates for \( K \) functionals. The main result of this paper is the following characterization of \( K(f, t, H_1, \text{BMO}) \).

**Theorem 3.1.** There exist constants \( c_1, c_2 > 0 \) depending only on \( n \) such that for all \( f \in H_1 + \text{BMO} \)
\[
(3.1) \quad c_1 t f_{H_1}^\#(t) \leq K(f, t, H_1, \text{BMO}) \leq c_2 t f_{H_1}^\#(t)
\]
In this section, we shall prove the lower estimate in (3.1); the upper estimate is proved in the next section.

The lower estimate in (3.1) rests on the behavior of \( f_{H_1}^\#(t) \) for \( t \) close to 0 and \( t \) close to \( \infty \).

**Lemma 3.2.** For any \( 1 < p < \infty \); there are constants \( c_1, c_2 > 0 \) such that
\[
(3.2) \quad c_1 f^\# \leq f_{H_1}^\# \leq c_2 f_{0,p}^\#
\]
in the sense that when one of these functions is finite the compared expression is also finite and smaller.

**Proof.** Suppose \( x \in \mathbb{R}^n \) and \( Q \) is any cube containing \( x \). There is a function \( \phi \in \mathcal{A}_Q \) with \( \phi \equiv 1 \) on \( Q \). Therefore,
\[
(3.3) \quad \frac{1}{|Q|} \int_Q |f - P_\phi f| \leq \frac{1}{|Q|} \int_Q |(f - P_\phi f)\phi| \leq \frac{c}{|Q|} \|(f - P_\phi f)\phi\|_{H_1} \leq c f_{H_1}^\#(x)
\]
Taking a sup over all cubes \( Q \) containing \( x \) in (3.3) and using (2.3) with \( p = 1 \) gives the lower estimate.

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For the upper inequality, we will use the fact that \( \| x^+ \|_{L_p^*} \leq c \| g \|_{L_p^*} \) for all \( g \in L_p \) (see [3]). Let \( Q \) again be a cube which contains \( x \) and let \( \phi \in \Phi_Q \). Then \( \psi := (f - P_\phi f) \phi \) is supported on \( \bar{Q} := 2AQ \). Using the estimates in (2.16(ii)) we have

\[
(3.4) \quad \| (f - P_\phi f) \phi \|_{H^1} \leq c \int_{\bar{Q}} \psi^+ \leq c \int_{Q} \psi^+ + c \frac{1}{|Q|} \psi^+
\]

\[
\leq c \left[ |Q| \left( \frac{1}{|Q|} \int_{Q} |f - P_\phi f|^p \right)^{\frac{1}{p}} + |Q| \int_{Q} \xi^p \right]
\]

\[
\leq c \left[ |Q| \left( \frac{1}{|Q|} \int_{Q} |f - P_\phi f|^p \right)^{\frac{1}{p}} + \xi^p \right]
\]

where the second to last inequality uses the fact that \( \int_{\bar{Q}} |x - z|^{-2n-2} c \). Since \( P_\phi \) is a bounded (with norms depending only on \( n \)) projection from \( L_p(Q) \) into \( P_N \), for any \( \pi \in P_N \), we can write \( f - P_\phi f = f - \pi - P_\phi (f - \pi) \) and find

\[
\frac{1}{|\phi|} \left( \int_{\bar{Q}} |f - P_\phi f|^p \right)^{\frac{1}{p}} \leq c \left( \int_{Q} |f - \pi|^p \right)^{\frac{1}{p}}
\]

Taking an inf over \( \pi \in P_N \) and using this back in (3.4) gives

\[
\frac{1}{|\phi|} \| (f - P_\phi f) \phi \|_{H^1} \leq c \left( \int_{Q} |f_0|^2 \right)^{\frac{1}{2}} \langle \xi \rangle + \xi \rangle
\]

Since \( f^\# \leq f_0^\# \) because of Hölder's inequality, we have the upper inequality in (3.2).

As a corollary to this lemma, we have

**Corollary 3.3.** There are constants \( c_1, c_2 > 0 \) such that for each \( f \in BMO \)

\[
(3.5) \quad c_1 \| f \|_{BMO} \leq \| f^\# \|_{L^\infty} \leq c_2 \| f \|_{BMO}
\]

**Proof.** This follows from (3.2) and the fact that \( \| f \|_{BMO} \approx \| f_0^\# \|_{L^\infty}, 1 \leq p < \infty \).

**Corollary 3.4.** If \( 1 < q < \infty \), there are constants \( c_1, c_2 > 0 \) such that for each \( f \in L_q \)

\[
\| f^\# \|_{H^1 L^q} \approx \| f \|_{L^q}
\]

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Proof. This follows from (3.2) by taking $1 < p < q$ in (3.2) applying $L_q$ norms and using the fact (see [1]) that $\| f^\# \|_{L_q} \approx \| f \|_{L_q}$. □

We also need an estimate for $f^\#_{H_1}$ near $H_1$.

Lemma 3.5. There is a constant $c$ such that for all $f \in H_1 + \text{BMO}$

\[
(3.6) \quad f^\#_{H_1}(x) \leq c M(E)(x)
\]

with $M$ the Hardy-Littlewood maximal operator.

Proof. Let $Q$ be any cube in $\mathbb{R}^n$, $x \in Q$ and $\phi \in \Phi_Q$. With $\psi := (f - P_\phi f)\phi$ and $\bar{Q} := 2AQ$, we have from (2.16)

\[
\| \psi \|_{H_1} \leq c \| \psi^+ \|_{L_1} \leq c \left( \int_a^{\bar{Q}} + \int_{\bar{Q}} \psi^+ \right) \leq c \left( \int \frac{\partial \tilde{f}}{\partial Q} + |Q| \tilde{f}^\#(x) \right)
\]

\[
\leq c |Q| \left( M(E)(x) + \tilde{f}^\#(x) \right) \leq c |Q| M(E)(x)
\]

where we used the fact that $\tilde{f}^\# \leq 2M(\tilde{f}) \leq 2M(E)$. Dividing by $|Q|$ and taking a sup over all $\phi \in \Phi_Q$ and $Q \ni x$ gives (3.6). □

Corollary 3.5. There is a constant $c$ such that for all $f \in H_1$ and $t > 0$

\[
(3.7) \quad t f^\#_{H_1}(t) \leq c \| f \|_{H_1}
\]

Proof. From (3.6), we have

\[
(3.8) \quad t f^\#_{H_1}(t) \leq ct M(E)^*(t) \leq c \| Mf \|_{L_1} \leq c \| f \|_{H_1}
\]

because $M$ is of weak type $(1,1)$. □

Proof of lower estimate in (3.1).

Suppose $f = b^g$ with $b \in H_1$ and $g \in \text{BMO}$. Since $P_\phi$ is a linear operator, it follows that $F + F^\#_{H_1}$ is sub-linear. Using this and (3.5), we have

\[
f^\#_{H_1} \leq b^\#_{H_1} + g^\#_{H_1} \leq b^\#_{H_1} + \| g^\#_{H_1} \|_{L_\infty}
\]

\[
\leq b^\#_{H_1} + c \| s \|_{\text{BMO}}, \quad \text{a.e.,}
\]

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Hence from Corollary 3.5,

\[ tf^{\#\#}_{H_1}(t) \leq t b^{\#\#}_{H_1}(t) + ct \|g\|_{BMO} \leq c(\|b\|_{H_1} + t \|g\|_{BMO}) \]

Taking now an inf over all such decompositions \( f = b + g \) gives the lower inequality in (3.1).

4. The upper estimate. To prove the upper estimate

\[ K(f,t,H_1,BMO) \leq ct f^{\#\#}_{H_1}(t) \]

We need to decompose \( f \) as \( f = b + g \) with \( b \in H_1 \) and \( g \in BMO \) satisfying

\[ \|b\|_{H_1} + t \|g\|_{BMO} \leq ct f^{\#\#}_{H_1}(t) \]

The decomposition we give is similar to that given in [4] for \( H_1 \) and \( L_\infty \).

Fix \( t > 0 \) and let \( E := \{x; f^{\#\#}_{H_1}(x) > f^{\#\#}_{H_1}(t)\} \). Then \( E \) is an open set with \( |E| \leq t \). Let \((Q_j)^\infty \) be a Whitney decomposition of \( E \) into dyadic cubes with \( d_j := \text{diam } (Q_j) \) and the usual properties [6, p. 167]:

(4.1) \[ i) \quad \bigcup_{j \in \mathbb{Z}} Q_j = E \]

\[ ii) \quad |Q_i \cap Q_j| = 0, \quad i \neq j. \]

\[ iii) \quad d_j \leq \text{dist } (Q_j,E^c) \leq 4d_j, \quad j=1,2,... \]

\[ iv) \quad \text{If } Q_i \text{ touches } Q_j, \text{ then } \text{diam } Q_i \leq 4 \text{ diam } Q_j. \]

\[ v) \quad \text{Any point } x \in E \text{ appears in at most } N_0 \text{ of the cubes } Q_j, \quad j=1,2,... \text{ with } N_0 \text{ depending only on } n. \]

If we let \( Q_j^* := \frac{3}{8} Q_j \), then there is a partition of unity \((\phi_j^*)\) (denoted by \((\phi_j^*)^\infty \) in [5]) subordinate to \((Q_j^*)\) with the properties

(4.2) \[ i) \quad \sum_{j \in \mathbb{Z}} \phi_j^* = 1 \text{ on } E \]

\[ ii) \quad \text{support of } \phi_j \text{ is contained in } Q_j^*, \quad j=1,2,... \]

\[ iii) \quad 0 \leq \phi_j \leq 1 \text{ and } \phi_j \equiv 1 \text{ on } \frac{3}{4} Q_j. \]

\[ iv) \quad \|D^\nu \phi_j\|_{L_\infty} \leq c d_j^{-|\nu|}, \quad \text{for all } \nu \text{ and } j. \]

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With the abbreviated notation $P_j := P_{\phi_j}$, we define our decomposition for $f$ by

$$g := \sum_{j=1}^{\infty} (P_j f) \phi_j + f \chi_{E_c} ; \quad b := \sum_{j=1}^{\infty} (f - P_j f) \phi_j$$

We begin by estimating the norm of $b$ in $H_1$. This is particularly simple since if $Q_j := 10 \sqrt{n} Q_j$ then $Q_j \cap E_c \neq \emptyset$ and therefore there is a point $x \in Q_j \cap E_c$. Since $\phi_j \in \Phi_{Q_j}$,

$$\| (f - P_j f) \phi_j \|_{H_1} \leq |Q_j| \int_{E_c} |x_j| f_{H_1}^\#(x_j) \leq c |Q_j| f_{H_1}^\#(t)$$

Adding these estimates gives

$$\| b \|_{H_1} \leq c \sum_{j=1}^{\infty} |Q_j| f_{H_1}^\#(t) \leq c |E| f_{H_1}^\#(t) \leq c t f_{H_1}^\#(t).$$

The estimate of the norm of $g$ in $\text{BMO}$ is somewhat more involved but similar to the Bennett-Sharples argument [1,56].

Lemma 4.1. We have $\| g \|_{\text{BMO}} \leq c f_{H_1}^\#(t)$ with $c$ depending only on $n$.

Proof. Let $Q$ be a cube in $E_c$. According to (3.2), it is enough to show that there is a constant $\alpha$ such that

$$\int_Q |g - \alpha| \leq c |Q| f_{H_1}^\#(t)$$

Let $A := \{ i : Q_i \cap Q \neq \emptyset \}$. We shall consider three cases.

Case 1. $A = \emptyset$. Then $Q \subset E_c$ and $g = f$ on $Q$, so we may take $\alpha = f_Q$ and the lower estimate in (3.2) to find

$$\int_Q |g - \alpha| \leq |Q| \inf_{Q} f_{Q}^\# \leq c |Q| \int_{E_c} f_{H_1}^\# \leq c |Q| f_{H_1}^\#(t)$$

because $f_{H_1}^\# \leq f_{H_1}^\#(t)$ on $E_c$.

Case 2. There is an $i \in A$ with $\text{diam}(Q) \leq \frac{1}{64} \text{diam}(Q_i)$ for some $i \in A$.

For the proof in this case we will use the fact that for each $h \in L_1(Q_i)$,
(4.7) \[ \left\| P_i(h) \right\|_{L^\infty(Q_i^*)} \leq \frac{c}{|Q_i^*|} \int_{Q_i^*} |P_i(h)| \leq \frac{c}{|Q_i^*|} \int_{Q_i^*} |h|, \ i = 1, 2, \ldots. \]

The first inequality in (4.7) is simply a comparison of polynomial norms (see e.g. [2, §3]) and the second is the boundedness of the projection \( P_i \) on \( L_1(Q_i^*) \) which in turn follows from (2.13).

Now let \( Q_{j_0} \) be a largest cube among the \( Q_i \) with \( i \in \Lambda \). Then

\[ Q \subseteq \frac{33}{32} Q_{j_0}^* \subseteq \frac{5}{4} Q_{j_0} \subset E. \]

If \( i \in \Lambda \), then \( \frac{5}{4} Q_i = Q_i^* \) intersects \( Q \) and hence intersects \( \frac{5}{4} Q_{j_0} \).

It follows from (4.1)iv that \( Q_i^* \) and \( Q_{j_0} \) touch and therefore \( Q_i^* \subset 4Q_{j_0} \).

Define \( \tilde{Q} := 10 \sqrt{n} Q_{j_0} \) and \( \alpha := \frac{\tilde{Q}}{Q} \). According to (4.2)ii) \( \tilde{Q} \cap E = \emptyset \) and therefore using (4.7) and (2.13) gives

\[ (4.9) \quad \int_{Q \cap E \setminus \tilde{Q}} = \int_{\bigcup_{i \in \Lambda \cap \tilde{Q} \cap E} P_i(\alpha \phi_i) \geq \sum_{i \in \Lambda \cap \tilde{Q} \cap E} \int_{Q_i \cap \tilde{Q}} |P_i(\alpha \phi_i)| \]

\[ \leq \sum_{i \in \Lambda \cap \tilde{Q} \cap E} \frac{|Q_i \cap \tilde{Q}|}{|Q_i^*|} \int_{Q_i^*} |f - \phi_i| \leq c \sum_{i \in \Lambda \cap \tilde{Q} \cap E} \frac{|Q_i \cap \tilde{Q}|}{|Q_i^*|} \int_{Q_i^*} |f - \phi_i| \]

\[ \leq c \left( \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \right) \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \leq c |\tilde{Q}| \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \leq c |\tilde{Q}| \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \]

where the third to last inequality uses the fact that \( |Q_i \cap \tilde{Q}| \geq \frac{1}{4} |Q_i^*| \),

because \( Q_i \) touches \( Q_{j_0} \); the second to last inequality uses the fact that \( |\tilde{Q}| \leq (10\sqrt{n})^n |Q_{j_0}| \); and the last inequality uses (3.2) and the fact that \( \tilde{Q} \cap E = \emptyset \).

**Case 3.** \( \Lambda \neq \emptyset \) and for all \( i \in \Lambda \), \( \text{diam}(Q_i) \leq 64 \text{diam}(Q) \). In this case \( Q_i \subset Q_i^* \subset 129Q \) for all \( i \in \Lambda \). Define \( \tilde{Q} := 1290 \sqrt{n} Q \). Since \( 10\sqrt{n} Q_i \) touches \( E \cap Q \) and is contained in \( \tilde{Q} \), we have \( \tilde{Q} \cap E \neq \emptyset \). We let \( \alpha := \frac{\tilde{Q}}{Q} \).

Using (4.7) and (4.1), we have

\[ (4.10) \quad \int_{Q \cap E \setminus \tilde{Q}} = \int_{Q \cap E \setminus \tilde{Q}} P_i(\alpha \phi_i) \geq \sum_{i \in \Lambda \cap \tilde{Q} \cap E} \int_{Q_i \cap \tilde{Q}} |P_i(\alpha \phi_i)| \]

\[ \leq \sum_{i \in \Lambda \cap \tilde{Q} \cap E} \frac{1}{|Q_i^*|} \int_{Q_i^*} |f - \phi_i| \]

\[ \leq c \left( \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \right) \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \leq c |\tilde{Q}| \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \leq c |\tilde{Q}| \inf_{Q} \left| \frac{\tilde{Q}}{Q} \right| \frac{1}{|Q|} \int_{Q \cap \tilde{Q}} |f - \phi_i| \]
\[ \leq c \sum_{i \in A} f_i \int_{Q_i} |f-\alpha| + \int_{Q_0} f - \alpha \leq c |Q_0| \inf_{Q_0} f^\theta \leq c |Q_0| \inf_{Q_0} f^\theta (\tau). \]

The three estimates (4.8–10) combine to prove (4.5).

Lemma 4.1 and the estimate (4.4) shows that \( f = b + g \) with

\[ \|b\|_{H^1} + \tau \|g\|_{BMO} \leq c \tau f^\theta (\tau) \]

which establishes the upper estimate in (3.1) and completes the proof of Theorem 3.1.

5. Acknowledgement. The problem of characterizing the \( K \) functional for \( (H^1, BMO) \) was posed to me by my colleagues, C. Bennett and R. Sharp. I thank them for this and various discussions concerning this work.

6. Postscript. I have recently heard that Björn Jawerth has also given a characterization of \( K(f, \tau, H^1, BMO) \) using the area integral. His paper has the same title as this paper and will appear in the Proceedings of the American Math. Soc.

References.

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