Week in Review, Sections 7.3-7.4

1. Find the volume of the solid obtained by rotating the region bounded by the $x$-axis and $y = (x-1)(x-3)$ about the $y$-axis.

$$V = 2\pi \int_{1}^{3} y(x) \sqrt{1 + y'(x)^2} dx$$

$$= 2\pi \int_{1}^{3} (x-1)(x-3) x \sqrt{1 + x^2} dx$$

$$= 2\pi \int_{1}^{3} x^2 (x^2 - 4x + 3) dx$$

$$= 2\pi \left[ \frac{x^4}{4} - 2\frac{x^3}{3} + \frac{3x^2}{2} \right]_{1}^{3}$$

$$= \frac{24\pi}{5}$$

2. Find the value of $a > 0$ such that when the area bounded by the curve $y = 1 + e^x$, the line $y = 1$ and the line $x = a$ is rotated about the line $y = 1$, a volume of $3\pi$ is generated.

$$V = 3\pi = 2\pi \int_{0}^{a} x e^{x^2} dx$$

$$\Rightarrow 3 = \pi \int_{0}^{a} e^{x^2} dx$$

$$\Rightarrow 3 = \frac{\sqrt{\pi}}{2} (e^a - 1)$$

$$\Rightarrow e^a = 4$$

$$\Rightarrow a = \sqrt{\ln 4}$$
3. Set up (BUT DO NOT INTEGRATE) the integrals, using shell method, to calculate

(a) the volume enclosed when the area between the curves \( y = \cos x \), \( y = 1 \) and \( x = \frac{\pi}{2} \) is rotated about the line \( y = 4 \).

\[
V = 2\pi \int_0^1 r \, dy = 2\pi \left[ (y-1)(\frac{\pi}{2} - \cos^{-1}(y)) \right]_0^1
\]

(b) the volume enclosed when the area between the curves \( y = \cos x \) and \( y = \sin x \) between \( x = -\frac{\pi}{4} \) and \( x = 3\pi/4 \) is rotated about the line \( x = -\pi/4 \).
4. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

\[ F = kx \]
\[ 40 = k \times 10 \Rightarrow k = \frac{40}{5} = 8 \text{ N/cm} \]

\[ F(x) = kx = 8x \]

\[ W = \int_{a}^{b} F(x) \, dx = \int_{5}^{8} 8x \, dx \]

\[ = 4x^2 \bigg|_{5}^{8} = 4(64 - 25) = 156 \text{ N.cm} \]

5. A 20 ft cable weighs 80 lbs and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.

\[ \lambda = \frac{80}{20} = 4 \text{ lb/ft} \text{ linear density} \]

will be lifted by \( 2x \) = distance

\[ \Delta W = F \cdot \text{distance} = (4 \Delta x)(2x) = 8x \Delta x \]

\[ \sum \Delta W \rightarrow W = \int_{0}^{10} 8x \, dx = 4x^2 \bigg|_{0}^{10} = 400 \text{ ft-lbs} \]
6. A man weighing 200 pounds climbs a ladder with a 100 pounds sack of sand that is leaking one pound per minute. If he climbs steadily at the rate of 5 feet per minute, and if the ladder is 40 feet high, then how much work does he do in climbing the ladder?

\[ W = \int_{0}^{40} F(y) \, dy \]

- **Weight of man + sack at level \( y \)**

\[ F(y) = 200 + \left( 100 - \frac{1}{5} (1) t \right) = 300 - \frac{y}{5} \]

\[ y = v \cdot t = 5t \]

\[ W = \int_{0}^{40} 300 - \frac{y}{5} \, dy = 300y - \frac{y^2}{10} \bigg|_{0}^{40} = 11,840 \text{ ft-lbs} \]

7. A man drags a 100 pound weight from \( x = 0 \) to \( x = 30 \) feet. He resists a wind which at position \( x \) applies a force of magnitude \( F(x) = x^3 + x + 40 \). How much work does he perform?

\[ W = \int_{0}^{30} x^3 + x + 40 \, dx \]

\[ = \frac{x^4}{4} + \frac{x^2}{2} + 40x \bigg|_{0}^{30} = \ldots 204,150 \text{ ft-lbs} \]
8. Find the work done in pumping all of the water from a small hole at the top of a full horizontal cylindrical tank of radius $r$ and length $\ell$. Water density is $1000 \text{ kg/m}^3$. 

**Hint:** Evaluate the integral by interpreting part of it as the area of the circle.

- **Horizontal slice:** rectangular (very thin)
  - **Area:** $\ell \, w$
  - **Volume:** $\ell \, w \, dy$
  - **Mass:** $\rho \ell \, w \, dy$
  - **Weight:** $\rho \ell \, w \, dy$

Distance layer: $r - y$

$$\Delta W = \left( \rho g \ell \, w \, dy \right) (r - y)$$

\[ W = \int_{-r}^{r} \rho g \ell \sqrt{r^2 - y^2} (r - y) \, dy \]

\[ = 2\rho g \ell \left[ \int_{-r}^{r} (r - y) \sqrt{r^2 - y^2} \, dy \right] = 2\rho g \ell \left[ \int_{0}^{r} \sqrt{r^2 - y^2} \, dy - \int_{-r}^{0} \sqrt{r^2 - y^2} \, dy \right] \]

\[ = \pi\rho g \ell \int_{0}^{r} \sqrt{r^2 - y^2} \, dy \]

\[ W = \pi\rho g \ell r^3 = 9800\pi \ell r^3 \]

**Remark:** $V = \pi r^2 \ell$

\[ \frac{W}{\text{weight}} = \frac{\pi \rho g \ell r^3}{\rho g (\pi r^2 \ell)} = \ell \Rightarrow W = \text{weight} \times \text{distance to the center of mass moves} \]
9. (a) Calculate the work required to pump all the water out of a spout 2 meters higher than the conical tank shown in the figure. Assume that the tank is full, distances are in meters, and the density of water is 1000 kg/m³.

\[
\text{distance layer} = 10 + 2 - y = 12 - y
\]

\[
\text{weight layer} = \rho g V = \rho g A \Delta y = \rho g (\pi r^2) \Delta y
\]

\[
r = \sqrt{\frac{y^2}{2}}
\]

\[
\Delta W = \text{distance layer} \times \text{weight layer} = (12 - y) \rho g \frac{\pi}{4} y^2 \Delta y
\]

\[
W = \int_0^{10} \frac{2100}{4} (12 - y) y^2 \, dy = \frac{\pi}{4} (18000) \left( \frac{y^4}{4} - \frac{y^6}{6} \right)_{10}^{10}
\]

\[
\leq 11,545,355
\]

(b) Now assume that the water is pumped out of the top of the tank. SET UP the integral (but do not integrate) to calculate the work required to pump all the water out of the tank.

\[
W = \int_0^{10} \frac{2100}{4} (10 - y) y^2 \, dy
\]
10. The shape of a reservoir is obtained by rotating the graph of $y = x^2$, between 0 and 4 meters, about the $y$-axis. What is the work done in pumping it full of water from a lake 3 meters below the bottom? Water density is 1000 kg/m³.

\[
W = \rho g \pi \int_{0}^{16} y^2 + 3y \, dy = \rho g \pi \int_{0}^{16} y^2 + 3y \, dy \\
= 1000 \pi \left( \frac{4^3}{3} + 3 \cdot 4^2 \right) \approx 53,857,789 \text{ J}
\]