Chapter 3 Section 6

Let $x_1(t)$ and $x_2(t)$ be two functions (typically, we're thinking of these as being two solutions of an ODE). Then their *Wronskian* is:

$$W[x_1, x_2](t) := \det \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} := x_1(t)x_2'(t) - x_1'(t)x_2(t).$$

Notice that if x_1 and x_2 are linearly independent, then $W[x_1, x_2](t) \neq 0$. Indeed if it did then:

$$\begin{aligned} x_1(t)x_2'(t) = x_1'(t)x_2(t) &\iff \frac{x_1'(t)}{x_1(t)} = \frac{x_2'(t)}{x_2(t)} \\ &\iff \frac{d}{dt}\log|x_1(t)| = \frac{d}{dt}\log|x_2(t)| \\ &\iff \log|x_1(t)| = \log|x_2(t)| + C \\ &\iff |x_1(t)| = A\,|x_2(t)|, \end{aligned}$$

from which we conclude that x_1 and x_2 are not linearly independent.

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There is a forumla for determining the solution to a non-homogeneous equation:

$$x''(t) + p(t)x'(t) + q(t)x(t) = g(t),$$

if we know two linearly independent solutions x_1 and x_2 of the CHE. If x_1 and x_2 are solutions to $x'' + px' + qx = 0$ then:

$$x_P(t) = -x_1(t) \int \frac{x_2(t)g(t)}{W[x_1, x_2](t)} dt + x_2(t) \int \frac{x_1(t)g(t)}{W[x_1, x_2](t)} dt.$$

This formula looks somewhat intimidating. But for us, p and q will typically by constant. So, if $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$ the Wronskian is just $(r_2 - r_1)e^{(r_1+r_2)t}$. Plugging this into the formula:

$$\begin{aligned} x_P(t) &= -e^{r_1 t} \int \frac{e^{r_2 t} g(t)}{(r_2 - r_1)e^{(r_1+r_2)t}} dt + e^{r_2 t} \int \frac{e^{r_1 t} g(t)}{(r_2 - r_1)e^{(r_1+r_2)t}} dt \\ &= -\frac{e^{r_1 t}}{r_2 - r_1} \int e^{-r_1 t} g(t) dt + \frac{e^{r_2 t}}{r_2 - r_1} \int e^{-r_2 t} g(t) dt, \end{aligned}$$

which is not as bad as the formula looked at first?

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Find the general solution to $x'' + 4x = 8\tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$. The solution to the CHE is $x_H(t) = c_1\cos(2t) + c_2\sin(2t)$. The Wronskian of the two functions is:

$$W[x_1, x_2](t) = 2\cos(2t)\cos(2t) - (-2)\sin(2t)\sin(2t) = 4.$$

The VOP formula is then:

$$\begin{aligned} x_p(t) &= -\cos(2t)\int \frac{1}{4}\sin(2t)\tan t dt + \sin(2t)\int \frac{1}{4}\cos(2t)\tan t dt \\ &= -\cos(2t)\int \frac{1}{2}\sin^2(t) dt + \sin(2t)\int \frac{1}{4}(2\cos t\sin t - \tan t) dt \end{aligned}$$