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Let f be a function defined on $[0, \infty)$. Its *Laplace Transform* is defined as the integral:

$$\mathcal{L}\{f\}(s) := \int_{t=0}^{\infty} f(t)e^{-st}dt = \lim_{b\to\infty} \int_{t=0}^{b} f(t)e^{-st}dt.$$

Observe that you put in a function of t and get out a function of s. That is, the Laplace Transform takes function of t and gives a function of s. Other notation is:

$$\mathcal{L}\{f\}(s) = F(s).$$

Since this is an improper integral, the limit might not converge. And so this is defined only when the limit in the definition of the improper integral converges.

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For which values of s does the integral

$$\int_{t=0}^{\infty} Ke^{at}e^{-st}dt$$

converge? Using the definition of improper integral:

$$\lim_{b\to\infty}\int_{t=0}^{b} Ke^{at}e^{-st}dt = \lim_{b\to\infty}\int_{t=0}^{b} Ke^{(a-s)t}dt = \frac{1}{a-s}K\left(e^{(a-s)b}-1\right).$$

Observe that if $a > s$ then this limit is infinite while if $a < s$, this limit is $\frac{1}{s-a}$. So it converges when $a < s$. Note that if $s = a$, the integrand is just K and this is infinite integral.
Based on this, we have the following theorem:

Theorem. Suppose that:

(1) f is piecewise continuous on $0 \le t \le A$ for any positive A and

(2) There is some $K \ge 0$ such that $|f(t)| \le Ke^{at}$ when $t \ge M$ for some M.

Then the Laplace transform $\mathcal{L}\{f\}(s)$ exists for all $s > a$.

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Find $\mathcal{L}\{1\}(s)$.
This is:

$$\mathcal{L}\{1\}(s) = \int_{t=0}^{\infty} e^{-st} dt = \frac{1}{s},$$

and this is valid for $s > 0$.
Find $\mathcal{L}\{e^{at}\}(s)$.
This what we did above, and it is just $\frac{1}{s-a}$ for $s > a$.

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Find $F(s)$ if:

$$f(t) = \begin{cases} 1 & 0 \leq t \leq a \\ 0 & a < t \end{cases}.$$

$$F(s) = \int_{t=0}^{\infty} f(t)e^{-st}dt = \int_{t=0}^{A} e^{-st}dt = \frac{1}{s}\left(1 - e^{-sA}\right).$$

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Find $\mathcal{L}\{\sin(at)\}(s)$. Integrating by parts twice:

$$F(s) = \lim_{b\to\infty} \int_{t=0}^{b} \sin(at)e^{-st}dt = \lim_{b\to\infty} \left(-\frac{e^{-st}(s\sin(at) + a\cos(at))}{s^2 + a^2}\right)\Big|_{t=0}^{b} = \frac{a}{s^2 + a^2},$$

for $s > 0$.

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Find the Laplace transform of $3e^{2t} - \sin(5t)$. sol Observe the more general fact:

$$\begin{aligned}\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) &= \int_{t=0}^{\infty} (c_1 f_1(t) + c_2 f_2(t)) e^{-st} dt \\ &= c_1 \int_{t=0}^{\infty} f_1(t) e^{-st} dt + c_2 \int_{t=0}^{\infty} f_2(t) e^{-st} dt \\ &= c_1 \mathcal{L}\{f_1\}(s) + c_2 \mathcal{L}\{f_2\}(s).\end{aligned}$$

Thus, based on our previous work, the Laplace transform is:

$$\frac{3}{s-2} - \frac{5}{s^2+25},$$

for $s > 2$.

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If we know $\mathcal{L}\{f\}(s)$ and $f(0)$, find $\mathcal{L}\{f'\}(s)$.

$$\mathcal{L}\{f'\}(s) = \int_{t=0}^{\infty} f'(t)e^{-st}dt.$$

Now we use IBP with $u = e^{-st}$ and $v' = f'$ so that $u' = -se^{-st}$ and $v = f$. So this is:

$$F(s) = f(t)e^{-st}|_{t=0}^{\infty} + s\int_{t=0}^{\infty} f(t)e^{-st}dt = -f(0) + sF(s).$$

Notice that for this to work, we needed to have $\lim_{b\to\infty} f(b)e^{-sb} = 0$ and this is something we assume if f is of exponential order.
Note that we can also compute $\mathcal{L}\{f''\}(s)$:

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0) = s\left(s\mathcal{L}\{f\}(s) - f(0)\right) - f'(0) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0).$$

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Find the solution to the IVP:

$$x''(t) - x'(t) - 2x(t) = 0, \qquad x(0) = 1, \quad x'(0) = 0.$$

We are going to take the Laplace Transform of both sides. We get:

$$\begin{aligned}
\mathcal{L}\{x''\}(s) - \mathcal{L}\{x'\}(s) - 2\mathcal{L}\{x\}(s) &= (s^2X(s) - sx(0) - x'(0)) - (sX(s) - x(0)) - 2X(s) \\
&= (s^2X(s) - s) - (sX(s) - 1) - 2X(s)) \\
&= X(s)(s^2 - s - 2) - s + 1 \\
&= 0.
\end{aligned}$$

Thus:

$$X(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}.$$

Now, we want to find a function whose Laplace transform is equal to this. Using partial fractions:

$$\begin{aligned}
\frac{s-1}{(s-2)(s+1)} &= \frac{2/3}{s+1} + \frac{1/3}{s-2} \\
&= \frac{2}{3}\mathcal{L}\{e^{-t}\}(s) + \frac{1}{3}\mathcal{L}\{e^{2t}\}(s) \\
&= \mathcal{L}\{\frac{2}{3}e^{-t}\}(s) + \mathcal{L}\{\frac{1}{3}\mathcal{L}\{e^{2t}\}(s) \\
&= \mathcal{L}\{\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\}(s).
\end{aligned}$$

There is a fundamental fact about the Laplace transform that says the following: if $\mathcal{L}\{x_1\}(s) = \mathcal{L}\{x_2\}(s)$ then $x_1(t) = x_2(t)$. Thus, we know that the Laplace transform of the solution to the IVP equals the Laplace transform of:

$$x(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t},$$

whence this is the solution to the IVP.