Chapter 6 Section 2 & 3

Find the solution to the IVP

$$x''(t) + x(t) = \sin 2t, \qquad x(0) = 2, \quad x'(0) = 1.$$

We start by taking Laplace transforms of both sides:

$$[s^2X(s) - sx(0) - x'(0)] + X(s) = \frac{2}{s^2+4} \iff [s^2+1]X(s) - 2s - 1 = \frac{2}{s^2+4}$$
$$\iff X(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1} = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}.$$

Now, we need to do a partial fraction decomposition of this:

$$\frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \iff 2s^3+3s^2+8s+6 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

Equating coefficients, we get:

$$\begin{aligned} s^3 &: 2 = A + C \\ s^2 &: 1 = B + D \\ s &: 8 = A + 4C \\ 1 &: 6 = B + 4D. \end{aligned}$$

The first equation says $A = 2 - C$, plug this into the third equation to get:

$$8 = (2 - C) + 4C = 2 + 3C \implies C = 2 \implies A = 2 - C = 2 - 2 = 0.$$

The second equation says $B = 1 - D$, plug this into the fourth equation:

$$6 = (1 - D) + 4D = 1 + 3D \implies D = \frac{5}{3} \implies B = 1 - D = 1 - \frac{5}{3} = -\frac{2}{3}.$$

So this means that:

$$X(s) = \frac{-\frac{2}{3}}{s^2+4} + \frac{2s+\frac{5}{3}}{s^2+1} = -\frac{1}{3}\frac{2}{s^2+4} + 2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1}.$$

Thus:

$$x(t) = \frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t.$$

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The unit step function is

$$u_c(t) = \begin{cases} 0 & t < c, \\ 1 & c \leq t \end{cases}.$$

This is a function that is “off” until $t = c$ and then “turns on” when at time $t = c$. So, for example, this could represent an external force in a non-homogeneous second order ODE that turns on at a certain time and stays on for the duration of the experiment.
Sketch the graph of $h(t) = u_{\pi}(t) - u_{2\pi}(t)$.

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Consider the function:

$$f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & 9 \leq t \end{cases}.$$

Let's try to do each piece. Note that the function that is 2 on $[0, 4)$ and is 0 everywhere else can be written as:

$$2(u_0(t) - u_4(t)).$$

So, we can write:

$$\begin{aligned} f(t) &= 2(u_0(t) - u_4(t)) + 5(u_4(t) - u_7(t)) - 1(u_7(t) - u_9(t)) + u_9(t) \\ &= 2u_0(t) + 3u_4(t) - 6u_7(t) + 2u_9(t) \\ &= 2 + 3u_4(t) - 6u_7(t) + 2u_9(t). \end{aligned}$$

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Find the Laplace transform of $u_c(t)$.

$$\mathcal{L}\{u_c\}(s) = \int_{t=c}^{\infty} e^{-st} dt = \frac{e^{-ct}}{s}.$$

Find the Laplace transform of f above. Using the linearity of the Laplace transform, this is:

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \mathcal{L}\{2 + 3u_4 - 6u_7 + 2u_9\}(s) \\ &= 2\mathcal{L}\{2\}(s) + 3\mathcal{L}\{u_4\}(s) - 6\mathcal{L}\{u_7\}(s) + 2\mathcal{L}\{u_9\}(s) \\ &= 2\frac{1}{s} + 3\frac{e^{-4s}}{s} - 6\frac{e^{-7s}}{s} + 2\frac{e^{9s}}{s}.\end{aligned}$$

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Assume that f is a function whose Laplace transform is defined. If

$$g(t) = \begin{cases} 0 & 0 \le t < c \\ f(t-c) & c \le t \end{cases}$$

Note that $g(t) = u_c(t)f(t-c)$. Find $\mathcal{L}\{g\}(s)$.
From the definition:

$$\begin{aligned} G(s) &= \int_{t=c}^{\infty} f(t-c)e^{-st}dt \\ &= \int_{t=0}^{\infty} f(u)e^{-s(u+c)}du \\ &= e^{-sc}\int_{t=0}^{\infty} f(u)e^{-su}du \\ &= e^{-sc}F(s). \end{aligned}$$

Note this implies:

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}.$$

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Find the inverse Laplace Transform of

$$F(s) = \frac{1 - e^{-2s}}{s^2}.$$

Note that $F(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$. From the table, we know that:

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\}(t) = t.$$

So:

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\}(t) = u_2(t)(t-2).$$

So:

$$f(t) = t - u_2(t)(t-2).$$

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A similar computation shows that:

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}(t).$$

Using this, find the inverse Laplace trans form of:

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

First, we need to complete the square:

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where

$$F(s) = \frac{1}{s^2 + 1}.$$

Note that $f(t) = \sin t$. Thus:

$$g(t) = e^{2t}\sin t.$$

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Find the inverse Laplace trans form of:

$$G(s) = \frac{s}{s^2 - 4s + 5}.$$

This is similar to the above. Note that we can write it as:

$$\frac{s}{s^2 - 4s + 5} = \frac{s-2}{s^2 - 4s + 5} + \frac{2}{s^2 - 4s + 5} = \frac{s-2}{(s-2)^2 + 1} + \frac{2}{(s-2)^2 + 1}$$

If $F(s) = \frac{s}{s^2+1}$, then the first term is $F(s-2)$. We know that $\mathcal{L}^{-1}\{F(s)\}(t) = \cos t$. So the inverse laplace transform of the first term is $e^{2t}\cos t$. So the inverse Laplace transform is:

$$e^{2t}\cos t + 2e^{2t}\sin t.$$

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Find the solution to the IVP

$$x'' - 4x' + 5x = 0, \qquad x(0) = 1, \qquad , x'(0) = 4.$$

Taking Laplace transforms of both sides:

$$[s^2 X(s) - s - 4] - 4[sX(s) - 4] + 5X(s) = 0$$

whence:

$$X(s) = \frac{s}{s^2 - 4s + 5}.$$

From the above work we know that $x(t) = e^{2t}\cos t + 2e^{2t}\sin t$.