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Let $f(t)$ and $g(t)$ be two functions whose Laplace transforms – denoted by $F(s)$ and $G(s)$ as usual – exist. Then $F(s)G(s) = \mathcal{L}\{h(t)\}$ where:

$$h(t) = \int_{\tau=0}^{t} f(t-\tau)g(\tau)d\tau = \int_{\tau=0}^{t} f(t)g(t-\tau)d\tau = (f*g)(t) = (g*f)(t).$$

The new function $f*g = g*f$ is called the *convolution of f and g*. Another way to write this is:

$$\mathcal{L}\{f*g\} = FG.$$

This is very similar to "multiplication" and has many similar properties:

(1) $f*g = g*f$

(2) $f*(g_1+g_2) = f*g_1 + f*g_2$

(3) $(f*g)*h = f*(g*h)$

(4) $f*0 = 0$

There are is also one thing that we don't have. Indeed, it is not true that:

$$f*1 = f.$$

To be clear, what I am saying is:

$$f*1 \neq f.$$

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Find the inverse Laplace transform of:

$$\frac{1}{s^2(s^2+1)}.$$

First, observe that this can be written as:

$$\frac{1}{s^2}+\frac{1}{s^2+1}=\mathcal{L}\{t\}(s)+\mathcal{L}\{\sin t\}(s)=\mathcal{L}\{\int_{\tau=0}^{t}(t-\tau)\sin(\tau)d\tau\}$$

So the inverse is just the integral. So we compute that now:

$$\begin{aligned}\int_{\tau=0}^{t}(t-\tau)\sin(\tau)d\tau &= (\tau-t)\cos(\tau)|_{\tau=0}^{\tau=t}-\int_{\tau=0}^{t}\cos(\tau)\\ &= t-\sin t.\end{aligned}$$

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Solve the IVP:

$$x'' + 4x = g(t), \qquad x(0) = 3, \qquad x'(0) = -1$$

Begin by taking Laplace Transforms of both sides:

$$X(s)(s^2 + 4) - 3s + 1 = G(s),$$

so

$$\begin{aligned} X(s) &= \frac{3s-1}{s^2+4} + G(s)H(s) \\ &= 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + G(s)H(s). \end{aligned}$$

where

$$H(s) = \frac{1}{s^2+4} = \mathcal{L}\{\frac{1}{2}\sin(2t)\}(s).$$

So, using the table and convolution:

$$\begin{aligned} x(t) &= 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_{\tau=0}^{t} \sin(2(t-\tau))g(\tau). \\ &= x_H(t) + x_P(t). \end{aligned}$$

Thus, using Laplace transforms allows us to split the solution into a piece that describes how the system behaves when there is no external force plus the system's response to an external force.

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