Chapter 5 Section 3

Recall this problem from last class:
Solve $(1-t)x''(t) + x(t) = 0$.
We did this before, but now we will try to do it by expanding at $t = 1$ – which is a singular point.
The form of the solution is:

$$x(t) = \sum_{k=0}^{\infty} a_k (t-1)^k.$$

Plugging this into the equation gives:

$$\begin{aligned}
0 &= (1-t)\sum_{k=2}^{\infty} k(k-1)a_k(t-1)^{k-2} + \sum_{k=0}^{\infty} a_k(t-1)^k \\
&= \sum_{k=2}^{\infty} k(k-1)a_k(t-1)^{k-1} + \sum_{k=0}^{\infty} a_k(t-1)^k \\
&= \sum_{k=1}^{\infty} (k+1)(k)a_{k+1}(t-1)^k + \sum_{k=0}^{\infty} a_k(t-1)^k \\
&= a_0 + \sum_{k=1}^{\infty} \left[(k+1)(k)a_{k+1} + a_k\right](t-1)^{k-1}.
\end{aligned}$$

Now, this implies that $a_0 = 0$ and that:

$$a_{k+1} = -\frac{1}{(k+1)k} a_k.$$

But if $a_0 = 0$ then $a_1 = 0$ and so on. And so all coefficients are zero. And so the solution this method comes up with is the identically zero solution which means this method doesn't work. The issue is that we tried to expand around a singular point.
In general, consider the IVP:

$$P(t)x''(t) + Q(t)x'(t) + R(t)x(t) = 0, \qquad x(0) = x_0, \qquad x'(0) = x_0.$$

Using standard Taylor coefficient things we know that in general $k!a_k = x^{(k)}(t_0)$ when we expand the solution $x(t)$ at t_0. If $P(t_0) \neq 0$, we can solve this for x'':

$$x''(t) = -\frac{Q(t)}{P(t)}x'(t) - \frac{R(t)}{P(t)}x(t),$$

in particular:

$$x''(t_0) = -\frac{Q(t_0)}{P(t_0)}x'(t_0) - \frac{R(t_0)}{P(t_0)}x(t_0).$$

Thus, we can solve for a_2. We can differentiate the equation:

$$x''(t) = -\frac{Q(t)}{P(t)}x'(t) - \frac{R(t)}{P(t)}x(t)$$

to get an expression for $x'''(t)$ and thus $x'''(t_0)$. As long as Q/P and R/P are analytic at the point t_0, we can continue doing this and get an expression for $x^{(k)}(t_0)$ and thus a_k.
This actually allows us to expand our study a little. Instead of requiring that P, Q, R are polynomials, we will require that they be "any" function. And in the equation:

$$P(t)x''(t) + Q(t)x'(t) + R(t)x(t) = 0,$$

we say that t_0 is an ordinary point if Q/P and R/P are analytic there. Otherwise, we say that t_0 is a singular point.

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Theorem. If t_0 is an ordinary point of the differential equation

$$P(t)x''(t) + Q(t)x'(t) + R(t)x(t) = 0,$$

that is if Q/P and R/P are analytic at t_0 then the general solution is:

$$x(t) = \sum_{k=0}^{\infty} a_k(t - t_0)^k = a_0x_1(t) + a_1x_1(t).$$

Here a_0 and a_1 are arbitrary and x_1, x_2 form a fundamental set of solutions. Furthmore, the radius of convergence of the series solutions x_1 and x_2 is at least as big as the minimum of the radius of convergence of Q/P and R/P.

When dealing with a rational function, the radius of convergence of the power series of $Q(t)/P(t)$ centered at t_0 is the distance from t_0 to the closest zero of P. BUT WARNING: we have to include ALL THE ZEROS - even the complex zeros of P!!

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Consider the equation:

$$(1-t^2)x'' - 2tx' + \alpha(\alpha+1)x = 0.$$

Find a lower bound for the radius of convergence for solutions expanded at $t = 0$. This is going to be the distance from 0 to the zeros of $P(t) = (1-t^2) = (1-t)(1+t)$. So the lower bound is 1.

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Consider the equation:

$$(1+t^2)x'' + 2tx' + 4t^2x = 0.$$

Find a lower bound for the radius of convergence for solutions expanded at $t=0$, $t=-\frac{1}{2}$.
This is going to be the distance from $t=0$ to the closest zero of $1+t^2$. What are the zeros of $P(t)$? They are $\pm i$.
The distance from $t=0$ to $\pm i$ is 1. So, for expanding the solution at $t_0=0$ the lower bound is 1.
For $-\frac{1}{2}$ we use the distance formula. The distance between $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.
So here this is:

$$\sqrt{(-\frac{1}{2}-0)^2+(0-1)^2} = \sqrt{5/4} = \frac{1}{2}\sqrt{5}.$$

So the IOC is at least as big as $(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$.