Chapter 7

Sometimes there are two quantities we are interested in and their derivatives depend on each other. (For example, predator-prey system, two tanks of water with salt, etc). Sometimes the equations can be written in the form

$$\begin{aligned} x_1'(t) &= ax_1(t) + bx_2(t) \\ x_2'(t) &= cx_1(t) + dx_2(t), \end{aligned}$$

where a, b, c, d are constants. An equation of this form is called an *first order system of linear equations.* Using matrix–vector notation we can write it as:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Recall that the definition of matrix–vector multiplication is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_1 \begin{pmatrix} a \\ c \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

If the matrix–vector notation stuff was just another way to write the system, it wouldn't be helpful. But, it allows us to use tools from linear algebra. Let's look at a specific example and see how we can use linear algebra.

Chapter 7

Let's find solutions to:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

There is a key observation. Observe that:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, the matrix times these vectors is just a multiple of that vector. Note that not all vectors have this property. For example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq \lambda \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

for any λ.

The second observation is that if $\begin{pmatrix} a \\ b \end{pmatrix}$ is any vector, then we can find two consants c_1, c_2 such that:

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We will discuss this more in a moment, but the reason this is true is essentially that this is the same as the system:

$$a = -c_1 + c_2$$
$$b = c_1 + c_2.$$

This is a system of two equations in two unknowns and so there (should) be a solution (note that the top and bottom part of the right hand side aren't multiples of each other). In particular, let $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution. Then:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1(t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Notice that since $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a different matrix for each value of t, the coefficients c_1 and c_2 have to change with t as well. Let's plug this into the ODE.

Chapter 7

Let's plug this into the ODE.

$$\begin{aligned}\left(c_1(t)\begin{pmatrix}-1\\1\end{pmatrix}+c_2(t)\begin{pmatrix}1\\1\end{pmatrix}\right)' &= c_1'(t)\begin{pmatrix}-1\\1\end{pmatrix}+c_2'(t)\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\left(c_1(t)\begin{pmatrix}-1\\1\end{pmatrix}+c_2(t)\begin{pmatrix}1\\1\end{pmatrix}\right)\\ &= c_1(t)\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}+c_2(t)\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &- c_1(t)\begin{pmatrix}-1\\1\end{pmatrix}+3c_2(t)\begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

This can be re–written as:

$$(c_1'(t)+c_1(t))\begin{pmatrix}-1\\1\end{pmatrix}+(c_2'(t)-3c_2(t))\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

These two vectors are linearly indpendent – meaning for us that they are not multiples of each other. What this implies then is that:

$$\begin{aligned}c_1'(t)+c_1(t)&=0\\ c_2'(t)-3c_2(t)&=0\end{aligned}$$

whence:

$$\begin{aligned}c_1(t)&=a_1e^{-t}\\ c_2(t)&=a_2e^{3t}.\end{aligned}$$

And so the solution to the ODE is:

$$\begin{pmatrix}x_1(t)\\x_t(2)\end{pmatrix}=a_1e^{-t}\begin{pmatrix}-1\\1\end{pmatrix}+a_2e^{3t}\begin{pmatrix}1\\1\end{pmatrix}.$$

The coefficients a_1 and a_2 are arbitrary and we can solve for them if we had an initial condition like:

$$\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}.$$

Chapter 7

Chapter 7

In general, if there are two numbers λ_1 and λ_2 and two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ with:

$$M\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1$$
$$M\boldsymbol{v}_2 7 = \lambda_2\boldsymbol{v}_2$$

and $\boldsymbol{v}_1 \neq c\boldsymbol{v}_2$, the general solution to the differential equation can be found by:

Chapter 7

Find the general solution to the ODE:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Given:

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Chapter 7

Find the general solution to the ODE:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}' = \begin{pmatrix} -.5 & 1 \\ -1 & -.5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Given:

$$\begin{pmatrix} -.5 & 1 \\ -1 & -.5 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = (-\frac{1}{2} + i) \begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$\begin{pmatrix} -.5 & 1 \\ -1 & -.5 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (-\frac{1}{2} - i) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$
.

Chapter 7