Chapter 7

In this lecture, we will learn how to find eigenvalues and eigenvectors of 2×2 matrices. There are two steps: finding the eigenvalues and finding the eigenvectors. First, we give some formal definitions.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix (the definition is formally the same for any $n \times n$ matrix). A scalar λ is called an eigenvalue if there is a 2 dimensional vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ that is not the zero vector with:

$$M\boldsymbol{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \lambda \boldsymbol{v}.$$

This can be written as:

$$\mathbf{0} = (M - \lambda I)\boldsymbol{v} = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives the system:

$$\begin{aligned} x(a - \lambda) + by &= 0 \\ xc + y(d - \lambda) &= 0. \end{aligned}$$

The second equation asserts that

$$x = -y\frac{d - \lambda}{c},$$

inserting this into the first equation gives:

$$0 = -y\frac{d - \lambda}{c}(a - \lambda) + by = \frac{y}{c}\left((d - \lambda)(a - \lambda) - bc\right).$$

If $y \neq 0$, then $(d - \lambda)(a - \lambda) - bc) = 0$. If $y = 0$ then $x \neq 0$ (since they can't both be zero by definition) and a similar argument shows that $(d - \lambda)(a - \lambda) - bc) = 0$ in this case, too. On the other hand, similar reasoning show that whenever $(d - \lambda)(a - \lambda) - bc) = 0$, there is a non–zero solution to $M\boldsymbol{v} = \mathbf{0}$. So, the eigenvalues are precisely the solution to:

$$(a - \lambda)(d - \lambda) - bc = 0.$$

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Find the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
We find the zeros of $p(\lambda) = (1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. So they are $\lambda_1 = -1$ and $\lambda_2 = 3$.

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Now, given an eigenvalue λ, how can we find the corresponding eigenvectors. They are just going to be the solutions to the equation $(M - \lambda I)\boldsymbol{v} = \mathbf{0}$. For a two dimensional system, this is just solving two linear equations in two unknowns.

Find the eigenvectors of $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

Beginning with $\lambda_1 = -1$ we want to solve:

$$\begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is just the system:

$$\begin{aligned} 2x + y &= 0 \\ 4x + 2y &= 0. \end{aligned}$$

Observe the second equation is just 2 times the first equation. So the first equation says $y = -2x$. This means that there is no restriction on x and so any vector of the form:

$$\begin{pmatrix} t \\ -2t \end{pmatrix}$$

is an eigenvector (provided $t \neq 0$ since then the vector would be the zero vector). In particular, to solve an ODE we'll need to pick one - typically we set t equal to something convinient. Here $t = 1$ is the case and so we'd say that $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector associated to the eigenvalue -1.

For $\lambda_2 = 3$ we have:

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is just the system:

$$\begin{aligned} -2x + y &= 0 \\ 4x - 2y &= 0. \end{aligned}$$

The second equation is -2 times the first. And the first equation says that $y = 2x$ and so the eigenvectors associated to 3 are those of the form $\begin{pmatrix} t \\ 2t \end{pmatrix}$. And so we can take $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as an eigenvector.

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Solve the ODE

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

From the above work:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

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Solve the equation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The characteristic polynomial is $(1-\lambda)(3-\lambda)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$. So, the only eigenvalue is $\lambda = 2$. So now we solve:

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives $x = -y$ and so the only eigenvectors are those of the form $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Notice that there is a problem since we need two linearly independent eigenvectors to do the analysis we talked about in the previous class.
To handle cases like this, we introduce the generalized eigenvector. The GEV is the solution to the equation $(M-\lambda)\boldsymbol{w} = \boldsymbol{v}$ where $\boldsymbol{v}$ is the eigenvector. Clearly $\boldsymbol{w}$ and $\boldsymbol{v}$ are linearly indpendent (otherwise, $\boldsymbol{w}$ is an eigenvector). So, we may write the solution to the ODE as:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1(t)\boldsymbol{v} + c_2(t)\boldsymbol{w}.$$

Plugging this into the equation:

$$c_1'(t)\boldsymbol{v} + c_2'(t)\boldsymbol{w} = c_1(t)M\boldsymbol{v} + c_2(t)M\boldsymbol{w} = c_1(t)\lambda\boldsymbol{v} + c_2(t)(\lambda\boldsymbol{w} + \boldsymbol{v}) = (c_1(t)\lambda + c_2(t))\boldsymbol{v} + c_2(t)\lambda\boldsymbol{w}.$$

This gives:

$$\begin{aligned} c_1'(t) &= \lambda c_1(t) + c_2(t) \\ c_2'(t) &= \lambda c_2(t). \end{aligned}$$

So, $c_2(t) = a_1 e^{\lambda t}$. Plugging this into the first equation gives:

$$c_1'(t) = \lambda c_1(t) + a_1 e^{\lambda t},$$

which is:

$$c_1'(t) - \lambda c_1(t) = a_1 e^{\lambda t}.$$

The IF if $\mu(t) = e^{-\lambda t}$ and this becomes:

$$(c_1(t)e^{-\lambda t})' = a_1,$$

whence:

$$c_1(t) = a_1 t e^{\lambda t} + a_2 e^{\lambda t}$$

and the general solution is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a_1 t e^{\lambda t}\boldsymbol{v} + a_2 e^{\lambda t}\boldsymbol{v} + a_1 e^{\lambda t}\boldsymbol{w} = a_2 e^{\lambda t}\boldsymbol{v} + a_1 e^{\lambda t}\left(t\boldsymbol{v} + \boldsymbol{w}\right).$$

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Now, we find the generalized eigenvector for the problem in question:

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This gives:

$$\begin{aligned} -x - y &= 1 \\ x + y &= -1. \end{aligned}$$

This gives $x + y = -1$ and hence $y = -x - 1$. And so the vectors that solve this are those of the form $\begin{pmatrix} t \\ -t-1 \end{pmatrix}$. Now, any non-zero vector here will work and so we can take $t = 0$. Thus $\boldsymbol{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. So the general solution is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right).$$

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Solve the ODE

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

We first find the eigenvalues. The polynomial is:

$$(-1-\lambda)^2 + 4 = (\lambda+1)^2 + 4.$$

So the eigenvalues are $\lambda_{1,2} = -1 \pm 2i$. Taking the + solution and finding the eigenvector, we get that the eigenvector is $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$. And so the solution is:

$$c_1 \Re(e^{-t+2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix}) + c_2 \Im(e^{-t+2it} \begin{pmatrix} 2i \\ 1 \end{pmatrix}) = c_1 e^{-t} \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix}.$$