Chapter 7

In this lecture, we will learn how to find eigenvalues and eigenvectors of 2×2 matrices. There are two steps: finding the eigenvalues and finding the eigenvectors. First, we give some formal definitions.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix (the definition is formally the same for any $n \times n$ matrix). A scalar λ is called an eigenvalue if there is a 2 dimensional vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ that is not the zero vector with:

$$M\boldsymbol{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \lambda \boldsymbol{v}.$$

This can be written as:

$$\mathbf{0} = (M - \lambda I)\boldsymbol{v} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives the system:

$$\begin{aligned} x(a-\lambda) + by &= 0 \\ xc + y(d-\lambda) &= 0. \end{aligned}$$

The second equation asserts that

$$x = -y\frac{d-\lambda}{c},$$

inserting this into the first equation gives:

$$0 = -y\frac{d-\lambda}{c}(a-\lambda) + by = \frac{y}{c}\left((d-\lambda)(a-\lambda) - bc\right).$$

If $y \neq 0$, then $(d-\lambda)(a-\lambda) - bc) = 0$. If $y = 0$ then $x \neq 0$ (since they can't both be zero by definition) and a similar argument shows that $(d-\lambda)(a-\lambda) - bc) = 0$ in this case, too. On the other hand, similar reasoning show that whenever $(d-\lambda)(a-\lambda) - bc) = 0$, there is a non–zero solution to $M\boldsymbol{v} = \mathbf{0}$. So, the eigenvalues are precisely the solution to:

$$(a-\lambda)(d-\lambda) - bc = 0.$$

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Find the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

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Now, given an eigenvalue λ, how can we find the corresponding eigenvectors. They are just going to be the solutions to the equation $(M - \lambda I)\boldsymbol{v} = \mathbf{0}$. For a two dimensional system, this is just solving two linear equations in two unknowns.

Find the eigenvectors of $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

Chapter 7

Solve the ODE

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

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Solve the equation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

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Solve the ODE

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$