Chapter 1 Section 2

We will continue to look at some of the examples from the previous class and we will actually learn how to solve them today. Recall the equation the modeled the field mouse population. It was:

$$\frac{dp}{dt} = \frac{p-900}{2}.$$

Notice that p is the unknown that we are solving for and it is a *function*. That is, finding a solution to a differential equation is similar to finding a solution to an algebraic equation, except now our unknown is a function and there is calculus involved. To solve this problem, we first rearrange:

$$\frac{1}{p-900}\frac{dp}{dt} = \frac{1}{2}.$$

The left hand side of this equation is really (by the chain rule) just:

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}.$$

By the Fundamental Theorem of Calculus, it follows that:

$$\ln|p(t)-900| - \ln|p(t_0)-900| = \int_{\tau=t_0}^{t} \frac{d}{d\tau}\ln|p(\tau)-900|\,d\tau = \int_{\tau=t_0}^{t}\frac{1}{2}d\tau = \frac{1}{2}t - \frac{1}{2}t_0 = \frac{1}{2}t + C.$$

In other words:

$$\begin{aligned} \ln|p(t)-900| = \frac{1}{2}t + C &\Leftrightarrow |p(t)-900| = e^{C}e^{\frac{1}{2}t} \\ &\Leftrightarrow p(t)-900 = \pm e^{C}e^{\frac{1}{2}t} \\ &\Leftrightarrow p(t) = Ae^{\frac{1}{2}t} + 900. \end{aligned}$$

For the first "$\Leftrightarrow$", we just took the exponential (base e) of both sides. For the second "$\Leftrightarrow$", we use the fact that $|p(t)-900| = \pm(p(t)-900)$. Then for the third "$\Leftrightarrow$" we just added 900 to both sides.
There are several comments to make here. First, we have shown that if $p(t)$ satisfies the given ODE, then it also satisfies $p(t) = A\exp\frac{1}{2}t + 900$. In other words, any solution to the given ODE is a function that has the form $A\exp\frac{1}{2}t + 900$. Examining things more closely, we have actually shown that any function of the form $p(t) = A\exp\frac{1}{2}t + 900$ is a solution of the given ODE. But this can also be verified directly:

$$\begin{aligned} \frac{d}{dt}(A\exp(\frac{1}{2}t) + 900) &= \frac{A}{2}\exp(\frac{1}{2}t) \\ &= \frac{A}{2}\exp(\frac{1}{2}t + 900 - 900) \\ &= \frac{(A\exp(\frac{1}{2}t) + 900) - 900)}{2} \\ &= \frac{p(t)-900}{2}. \end{aligned}$$

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Second, this gives a family of solutions rather than a single solution. That is, the ODE *alone* cannot determine a unique solution. If you think back to the direction field, this makes sense. For every starting population – $p_0 = p(0)$ – there was an integral curve that passed through the point $(0, p_0)$. But all the integral curves are graphs of functions that satisfy the ODE so there must be infinitley many functions that satisfy the given ODE. We can determine A if we have some additional piece of information, usually referred to as an *initial condition*. Indeed, if we want a function that satisfies:

$$\frac{dp}{dt} = \frac{p - 900}{2}, \qquad p(0) = p_0, \tag{1}$$

then we just solve:

$$p_0 = A\exp(\frac{1}{2}0) + 900,$$

and see that $A = p_0 - 900$. So, the solution to (1) is $p(t) = (p_0 - 900)\exp(\frac{1}{2}t) + 900$. The problem (1) is called an *initial value problem*. We can also specify other conditions:

$$\frac{dp}{dt} = \frac{p - 900}{2}, \qquad p(t_0) = p_0. \tag{2}$$

This is also called an *initial value problem* even when t_0 isn't the "initial" t value. We will see later that problems (1) and (2) always have a solution (existence) and only one solution (uniqueness).

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The third comment to make is that when $A = 0$, the solution becomes $p(t) \equiv 900$. Notice that this is a zero of the right hand side of $\frac{dp}{dt} = \frac{p-900}{2}$. In general, if :

$$\frac{dx}{dt} = f(x),$$

and $f(x_0) = 0$, then $x(t) \equiv x_0$ is a solution. This is because $\frac{d}{dt}x_0 = 0$ (since x_0 is a constant) and $f(x_0) = 0$, so both sides of the equation are 0. We will discuss this more later.
Also, for reasons that will be clarified later, the equation:

$$\frac{dp}{dt} = \frac{p}{2},$$

is called the *corresponding homogeneous equation.* The function $p_H(t) = A\exp(\frac{1}{2}t)$ is a solution to this equation. This is important for the following reason (stated in a slightly more general formulation), if $x_P(t)$ is a particular solution of:

$$\frac{dx}{dt} = f(t)x(t) + g(t),$$

then any solution to this equation can be written in the form:

$$x(t) = x_H(t) + x_P(t),$$

where $x_H(t)$ is a solution to the corresponding homogeneous equation:

$$\frac{dx}{dt} = f(t)x(t).$$

In the current example, this means that every solution to

$$\frac{dp}{dt} = \frac{p-900}{2}$$

can be written as:

$$p(t) = p_H(t) + p_P(t) = A\exp(\frac{1}{2}t) + 900.$$

Of course, we already knew this, but this gives a different way to find the solutions (and this will be especially relevant when discussing second order equations.)

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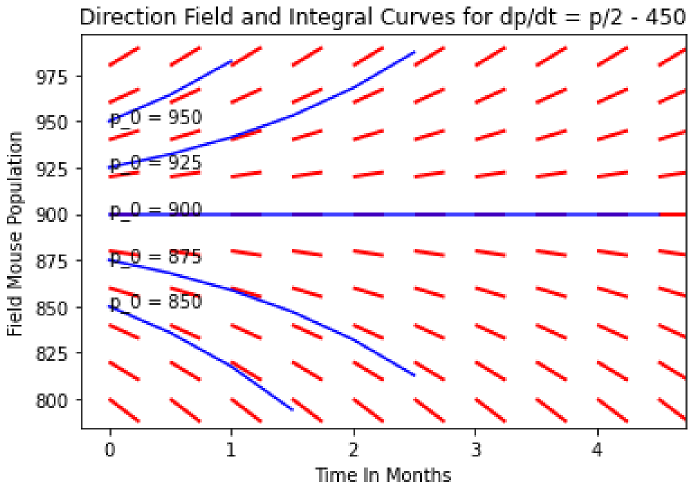
Below, there is a plot of the direction field with some integral curves plotted as well.



The blue curves represent integral/solution curves to the ODE. Notice how they "follow along" the direction field, but not perfectly. If we had sampled at more places to make the direction field, the curves would have been even closer to matching the direction field.

Notice also that solutions that start near $p_0 = 900$ move away from this solution very quickly. This should cause us to question the validity of our model for populations that start very far from this value. We will see later that the solution $p(t) \equiv 900$ is an example of an "unstable equilibrium".

Chapter 1 Section 3

There are many different types of differential equations. The type of equation you are dealing with will inform which techniques you will use. There is no general method for all differential equations, and so this is important. The first classification is based on the number of independent variables. If there is one independent variable, the equation is called a(n) *ordinary differential equation.* If there is more than one independent variable, the equation is called a(n) *partial differential equation*. This class deals with the first type. A an example, the following is an example of a *partial differential equation* (note that the derivatives are partial derivatives):

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}.$$

The next classification depends on the number of unknowns. So far, the equations we have looked at involve one unknown function. But we will see examples that involve 2 or more unknowns. For example:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_2(t) \\ x_2'(t) &= x_1(t)x_2(t). \end{aligned}$$

is a system of *two equations* in *two unknowns.*
Next we classify equations based on the order of the highest derivative. For example:

$$\begin{aligned} x''(t) + x(t) &= 0 && \text{second order;} \\ x'(t) + x(t) &= 3t && \text{first order;} \\ \cos(x^{(4)}(t)) + x''(t)x'(t) &= x(t) && \text{fourth order.} \end{aligned}$$

One of the most important classifications is linearity. A differential equation $f(t, x, x', \ldots, x^{(n)}) = 0$ is called linear if f is linear in the variables $x, x', \ldots, x^{(n)}$. That is, it has the form:

$$f_0(t) + f_1(t)x(t) + f_2(t)x'(t) + \cdots + f_n(t)x^{(n)}(t) = 0.$$

Another way to express this is the following: The differential equation $f(t, x, x', \ldots, x^{(n)}) = 0$ is linear if and only if whenever $x_1(t), x_2(t)$ are solutions then $c_1x_1(t) + c_2x_2(t)$ is a solution (here c_1, c_2 are constants).

$$\begin{aligned} x''(t) + x'(t) + x(t) &= 0 && \text{linear;} \\ (\cos t)x''(t) + e^t x'(t) + t^2 x(t) &= t && \text{linear;} \\ \cos(x''(t)) + x'(t) + x(t) &= 0 && \text{not linear;} \\ \theta''(t) + \frac{g}{L}\sin(\theta(t)) &= 0 && \text{not linear.} \end{aligned}$$

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Consider the general n^{th} order equation:

$$f(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0.$$

A function $\varphi(t)$ is a *solution* to this equation over an interval $\alpha < t < \beta$ if:

$$f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n)}(t)) = 0,$$

for all t in the interval (α, β).
For example, consider the equation $x''(t) + x(t) = 0$. The function $x(t) = \cos t$ is a solution to this equation since:

$$(\cos t)'' + \cos t = (-\sin t)' + \cos t = -\cos t + \cos t = 0.$$

There are three important questions that I alluded to earlier. Given a differential equation:

(1) Does it have at least one solution? (Existence)

(2) Does it have at most one solution? (Uniqueness)

(3) Can we find a solution to the differential equation? If not, what can we determined about the solution?