Chapter 7

Today, we are going to do some *qualitative analysis* of systems of ODEs. To understand what this means, it is best to proceed by means of an example.
Consider the system:

$$\boldsymbol{x}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{x}(t).$$

Sketch some solution curves in the (x_1, x_2) plane. First, the eigensystem is:

$$\lambda_1 = -1; \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$\lambda_2 = 3; \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This means all solutions are of the form:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Consider an initial value with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. This will imply $c_2 = 0$ and $c_1 = 1$. In other words, the solution stays on the vector line that is parallel to $\boldsymbol{v}_1$. And, as $t \to \infty$ the solution goes to the origin. Similarly, if if $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then we see that the solution stays on the line parallel to $\boldsymbol{v}_2$ and goes to "infinity" as $t \to \infty$ (here, going to infinity means that the the magnitude of the solution goes to infinity.)
For the solutions that are linear combinations with neither c_1 nor c_2 equal to zero, we analyze in the following way. For large positive values of t, the solution will approach $c_2 e^{3t} \boldsymbol{v}_2$. For large negative values of t, the solution will approach $c_1 e^{-t} \boldsymbol{v}_1$.

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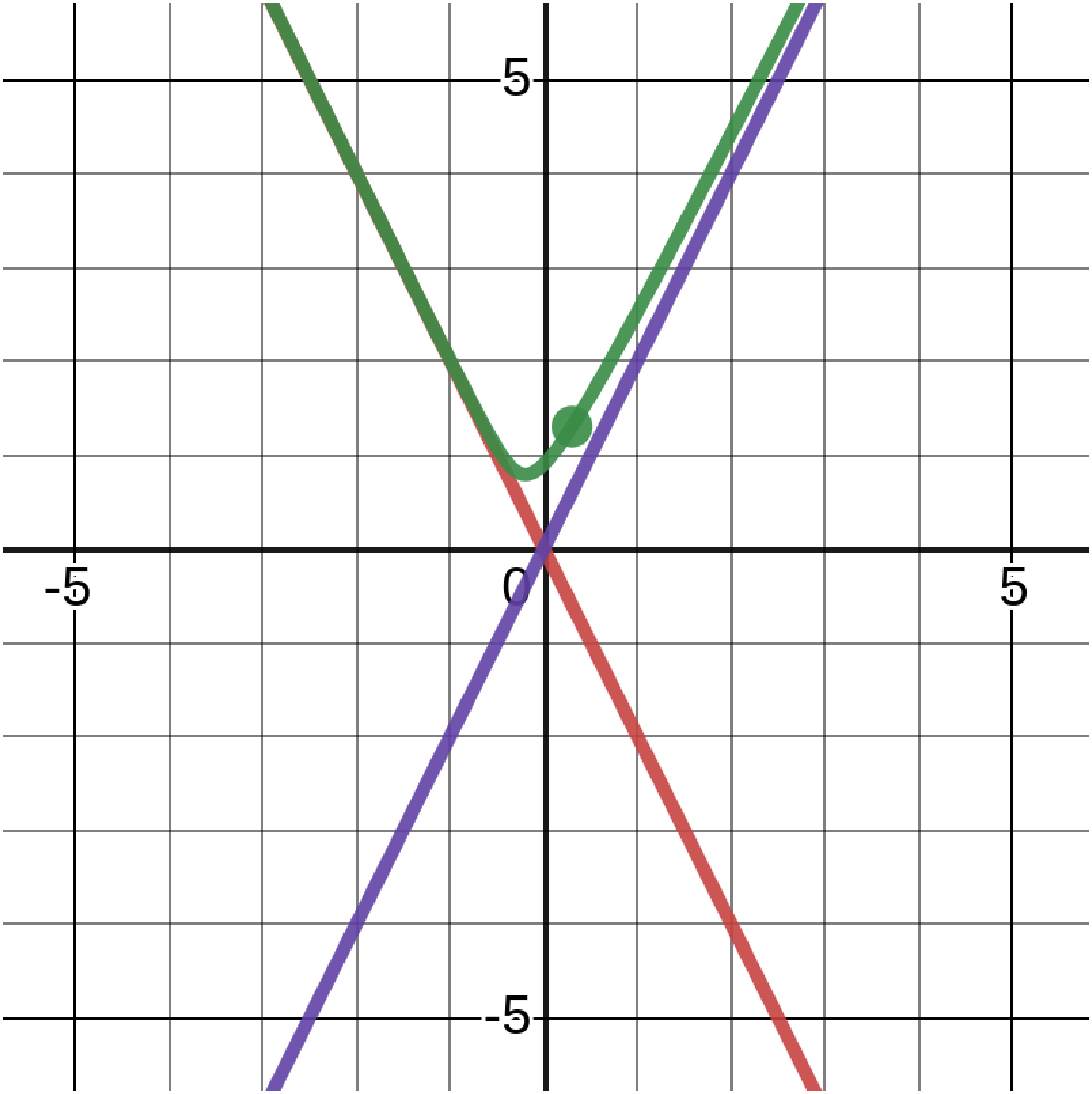

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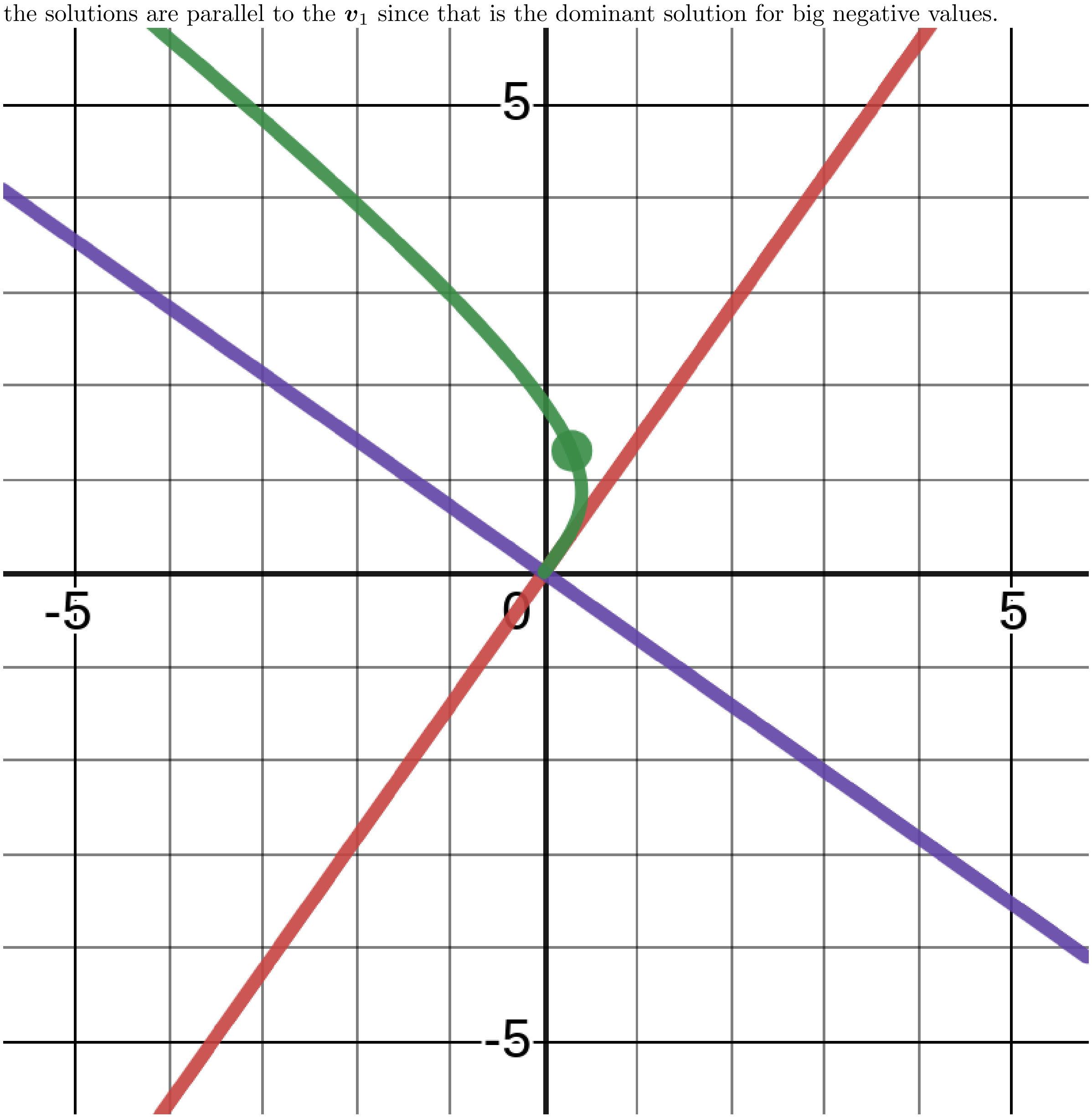
Do the same analysis with the system:

$$\boldsymbol{x}'(t) = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \boldsymbol{x}(t).$$

The eigensystem is:

$$\lambda_1 = -4, \boldsymbol{v}_1 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \qquad \lambda_2 = -1, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$

Note that since both eigenvalues are negative, the solutions go to zero as $t \to \infty$. On the other hand, as $t \to \infty$, the solutions are parallel to the $\boldsymbol{v}_1$ since that is the dominant solution for big negative values.



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Do the same analysis with the system:

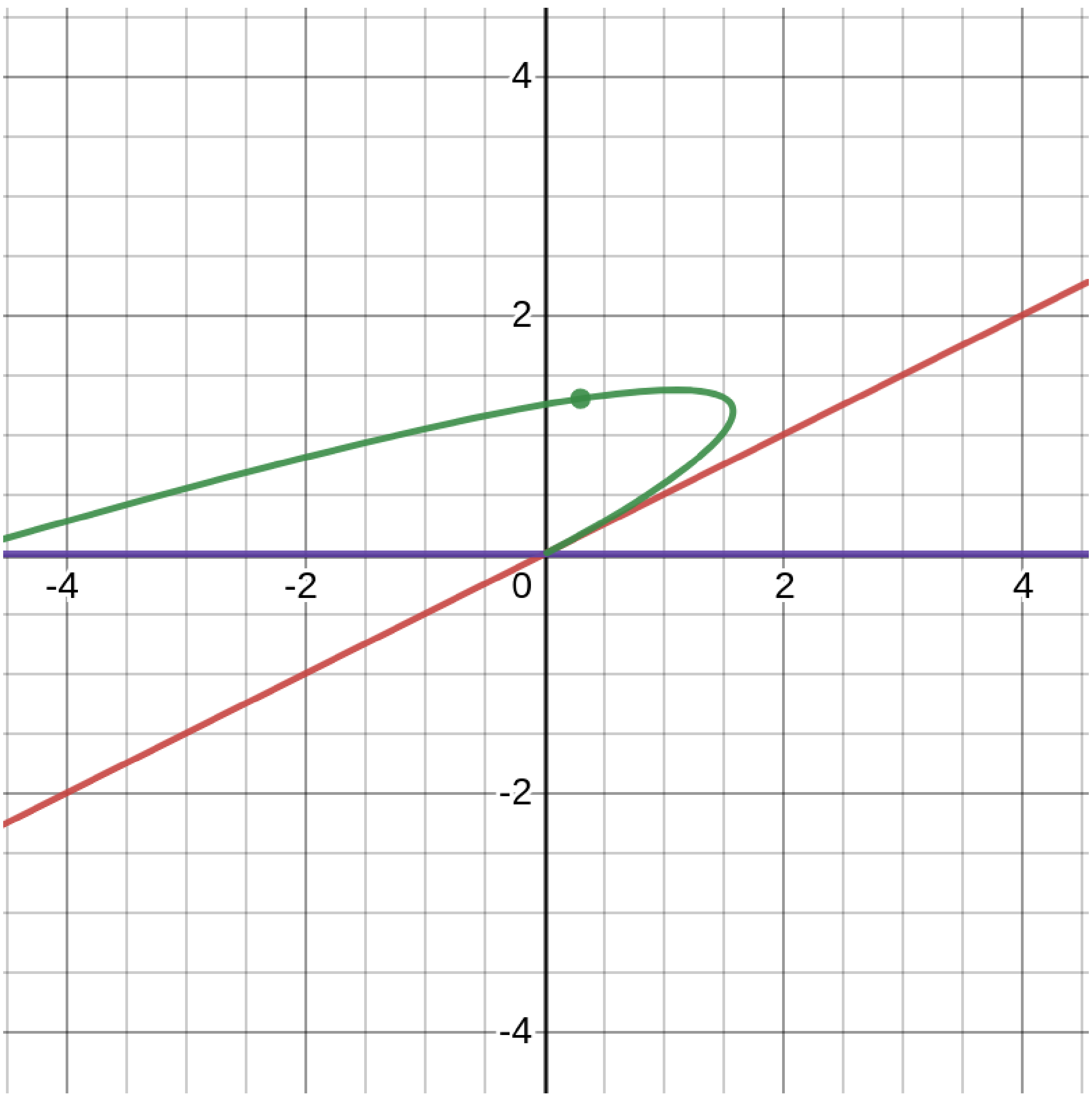
$$\boldsymbol{x}'(t) = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \boldsymbol{x}(t).$$

The eigensystem is:

$$\lambda_1 = 1, \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

So, there is only one eigenvector. The generalized eigenvector is $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general solution is:

$$c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^t \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$



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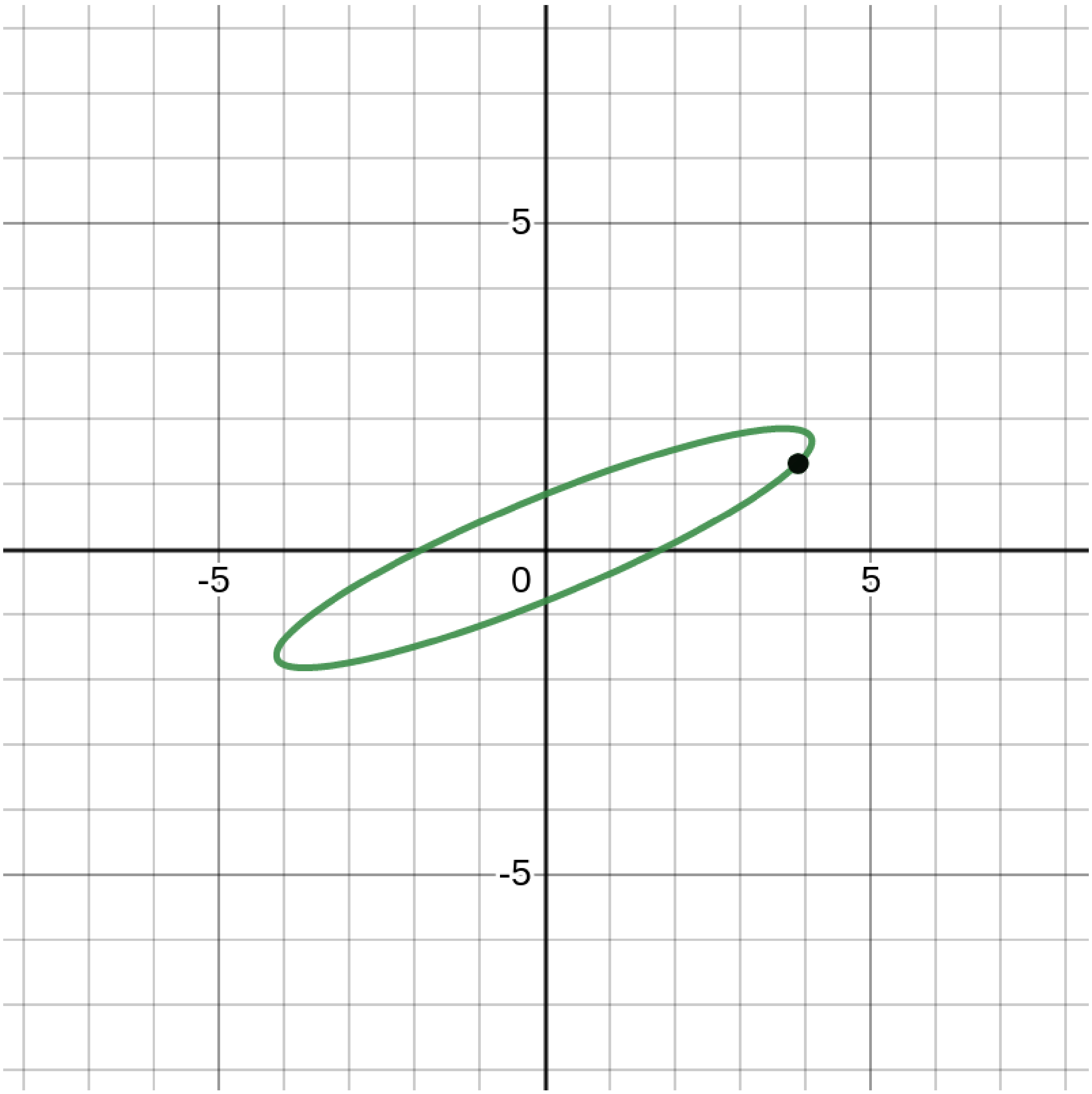
Do the same analysis with the system:

$$\boldsymbol{x}'(t) = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \boldsymbol{x}(t).$$

The general solution is:

$$c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}.$$



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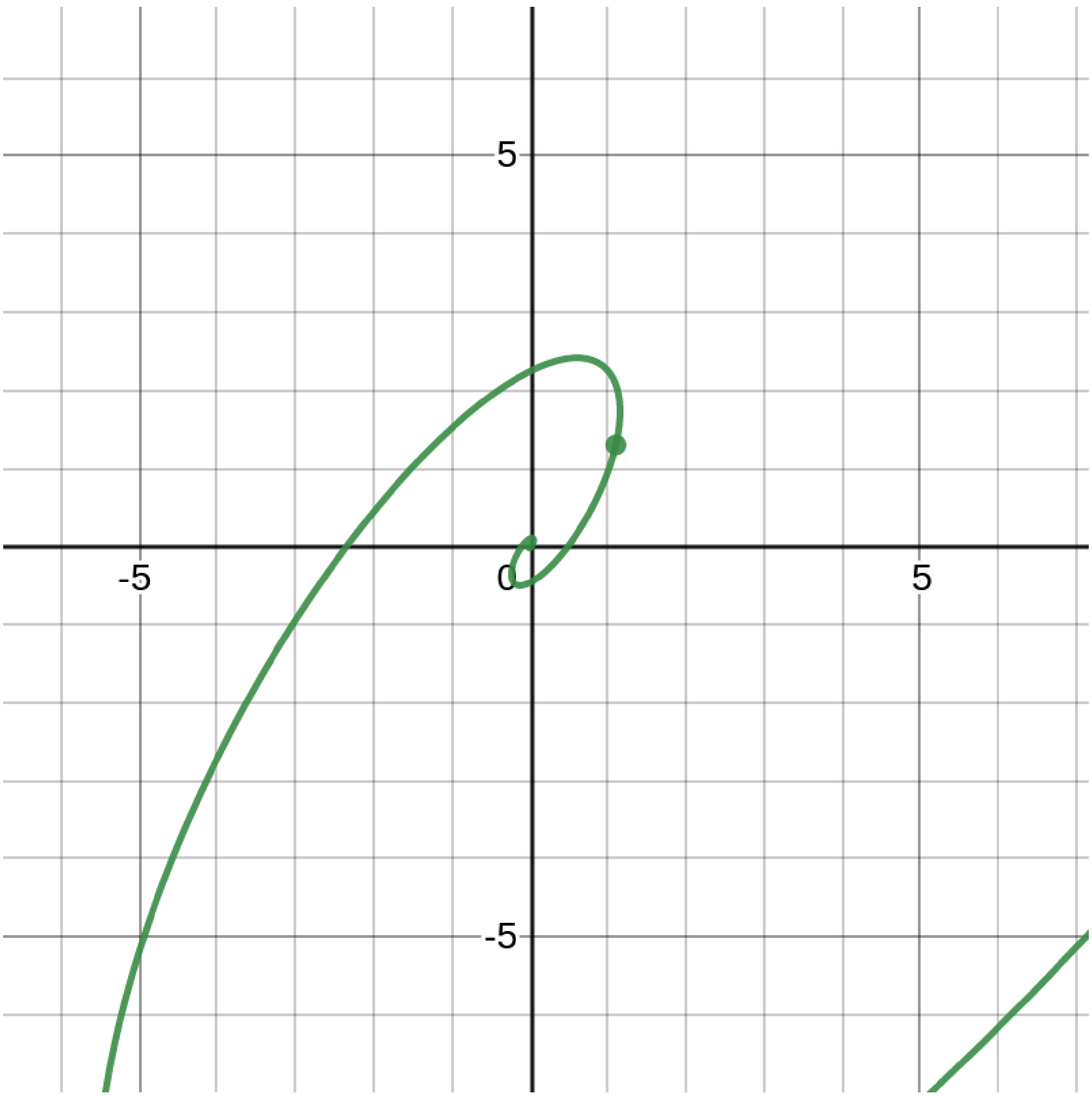
Do the same analysis with the system:

$$\boldsymbol{x}'(t) = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \boldsymbol{x}(t).$$

The general solution is:

$$c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{pmatrix}.$$



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