## Chapter 9

Consider the non–linear ODE:

$$x' = f(x) = x(x-1).$$

This is a non–linear equation and these are more difficult to solve than linear equations. This particular equation is not too hard to solve, but we will outline some general principles here. Notice that the equilibrium solutions are the ones that correspond to the zeros of $f(x)$, that is: $x(t) \equiv 0$ and $x(t) \equiv 1$. Notice also that at any point $x_0$ we can approximate $f(x)$ with its derivative:

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0),$$

and this is valid as long as $x$ is close to $x_0$. In addition, notice that if $f(x_0) = 0$, this equation is simpler. Indeed, if $x_0 = 0$ then this becomes:

$$f(x) \simeq f'(0)(x)$$

and if $x_0 = 1$ this becomes:

$$f(x) \simeq f'(1)(x-1).$$

Thus, if $x(t)$ is a solution to the ODE, then for $x(t)$ close to 0, its derivative can be approximated as:

$$x'(t) = f(x) \simeq f'(0)x$$

and if $x(t)$ is a solution to the ODE that is close to 1, then its derivative can be approximated as:

$$x'(t) = f(x) \simeq f'(1)(x-1).$$

Below, we will approximate the differential equation at the equilibrium points. That is, we will solve the equation $x' = f'(x_0)(x - x_0)$. The solution to those equations will approximate the actual solutions as long as they are near the equilibrium solutions. This is also used to determine stability of the equilibrium solutions. If the approximate solutions remain close to the equilibrium solutions, then that equilibrium is stable; if the approximate solutions go away from the equilibrium solutions, then that equilibrium is unstable. First, note that $f'(x) = 2x - 1$.

Linearization near $x_0 = 0$:
The linearized equation is $x' = (-1)(x) = -x$. The solution to this equation is $x(t) = x(0)e^{-t}$. This means that *if $x(t)$ is close to* 0 that the solution to $x' = x(x-1)$ is approximated by $x(0)e^{-t}$. What this indicates is that if $x(0)$ is close to 0, then $x(t) \simeq x(0)e^{-t}$ and this approximation is valid as long as $x(t)$ stays close to 0. But note that $x(0)e^{-t}$ gets closer and closer to 0 as $t \to \infty$ and so this approximation remains valid for all $t$. In other words, since the approximation gets closer and closer to the equilibrium solution as $t \to \infty$, we say that $x(t) \equiv 0$ is a *stable* equilibrium (which is consistent with what we got when we did the phase plane stuff in Chapter 2).

Linearization near $x_0 = 1$:
The linearized equation is $x' = (2(1)-1)(x-1) = x-1$. The solution to this equation is $x(t) = (x(0)-1)e^{t} + 1$. This means that *if $x(t)$ is close to* 1 that the solution to $x' = x(x-1)$ is approximated by $(x(0)-1)e^{t} + 1$. What this indicates is that if $x(0)$ is close to 1, then $x(t) \simeq (x(0)-1)e^{t} + 1$ and this approximation is valid as long as $x(t)$ stays close to 1. But note that $(x(0)-1)e^{t}+1$ gets farther and farther from 1 as $t \to \infty$ and so this approximation becomes worse as $t \to \infty$. In other words, since the approximation gets farther and farther from the equilibrium solution as $t \to \infty$, we say that $x(t) \equiv 1$ is an *unstable* equilibrium (which is consistent with what we got when we did the phase plane stuff in Chapter 2).

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In two dimensions, we can do something similar. Let's say we have a non-linear system:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \boldsymbol{F}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

where $f$ and $g$ are nonlinear. The only difference is that the derivative of $\boldsymbol{F}$ is the $2 \times 2$ matrix:

$$\boldsymbol{F}(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

and so the linear approximation reads:

$$\boldsymbol{F}(x,y) \simeq \begin{pmatrix} f(x_0,y_0) \\ g(x_0,y_0) \end{pmatrix} + \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Similar to the above, note that if $\boldsymbol{F}(x_0,y_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then this becomes:

$$\boldsymbol{F}(x,y) = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

At each point where $\boldsymbol{F}(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we can linearize this equation, solve that linear equation, and use that to give some information about the solutions to the actual equation.
We will illustrate this with the equation:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \boldsymbol{F}(x,y) = \begin{pmatrix} -(x-y)(1-x-y) \\ x(2+y) \end{pmatrix}.$$

So, the derivative matrix is:

$$\boldsymbol{F}'(x,y) = \begin{pmatrix} 2x-1 & 1-2y \\ y+2 & x \end{pmatrix}.$$

Now we need to find the places where $\boldsymbol{F}(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This involves solving a system of non–linear equations. That is:

$$\begin{pmatrix} -(x-y)(1-x-y) \\ x(2+y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solving this, we get four solutions: $(0,0)$; $(0,1)$; $(-2,-2)$; $(3,-2)$. And so we will linearize at each of these points. The work–flow is as follows. First we linearlize the equation to get:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \simeq \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Then, we solve the associated equation:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \simeq \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and sketch the solution trajectories (these will be near $(0,0)$). Then we "copy and paste" those trajectories to the equilibrium solution $(x_0,y_0)$.

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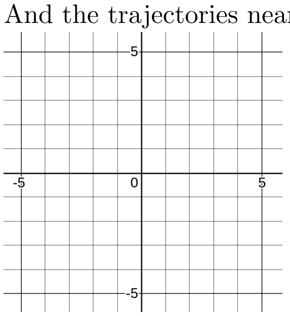
Linearization at $(0, 0)$:

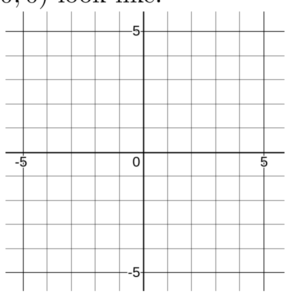
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigensystem is:

$$\lambda_1 = -2; \qquad \boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda_1 = 1; \qquad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Since the eigenvalues are opposite sign, this tells us that the stability of equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is *saddle point.* And the trajectories near $(0, 0)$ look like:



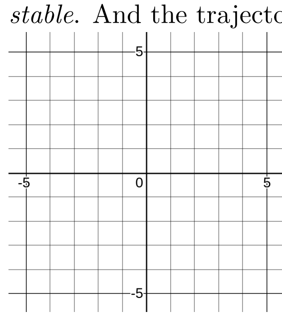

Linearization at $(-2, -2)$:

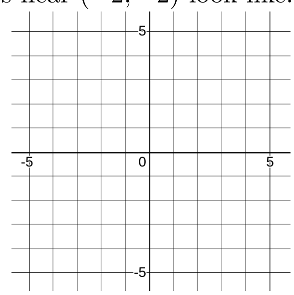
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigensystem is:

$$\lambda_1 = -5; \qquad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_1 = -2; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}.$$

Since the eigenvalues are negative, this tells us that the stability of equilibrium solution $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is *asymptotically stable.* And the trajectories near $(-2, -2)$ look like:



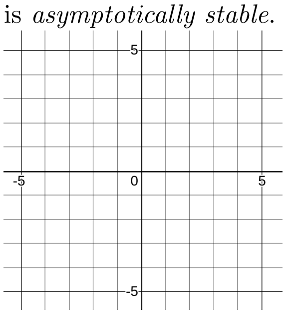

Linearization at $(0, 1)$:

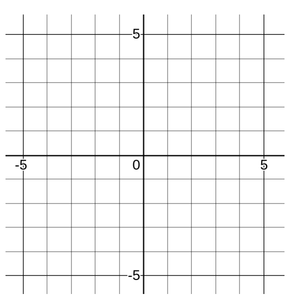
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigensystem is:

$$\lambda_{1,2} = \frac{1}{2}(-1 + i\sqrt{11}).$$

The eigenvalues are complex with negative real part. So the trajectories are spirals. By plugging $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into the ODE we see that spirals are counter–clockwise. In addition, this implies that the equilibrium solution $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is *asymptotically stable.*



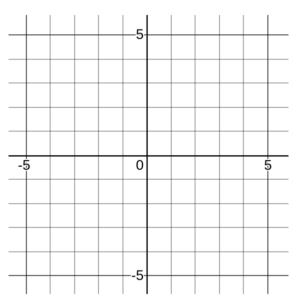

Linearization at $(3, -2)$:

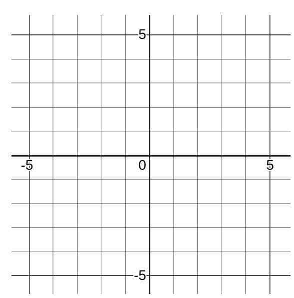
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigensystem is:

$$\lambda_1 = 5; \qquad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \lambda_1 = 3; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

Since the eigenvalues are positive, this tells us that the stability of equilibrium solution $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is *unstable.* And the trajectories near $(3, -2)$ look like:



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And here is what they look like when all drawn on the same phase plane: