Chapter 2 Sections 1 & 2

Chapter 2 of the book deals with first order equations of the form:

$$\frac{dx}{dt} = f(t, x),$$

where $f(t, x)$ is a *known* function of two variables. There is not a practical, universal method that can be used to solve all equations of this class. However, there are important subclasses for which there are methods. We will deal with *linear* and *separable* equations today. Later, we will learn about *exact* equations.

Linear Equations

The general form of a first order liner differential equation is:

$$A(t)x'(t) + B(t)x(t) = c(t).$$

Dividing by A, we get the *standard form*

$$x'(t) + b(t)x(t) = g(t).$$

We need to be careful about dividing by A (what if it is 0?), and we will deal with that a little later in the course. For now, just note that on any interval on which A is never 0, we can do this.
When $g \equiv 0$, we say this is a *first order, linear, homogeneous equation.* In that case, the equation reduces to:

$$x'(t) = -b(t)x(t) \quad \Leftrightarrow \quad \frac{x'(t)}{x(t)} = -b(t) \quad \Leftrightarrow \quad \frac{d}{dt}(\log x(t)) = -b(t)$$

In other words:

$$\log x(t) - \log x(t_0) = -\int_{\tau=t_0}^{t} b(\tau)d\tau \quad \Leftrightarrow \quad x(t) = \exp\{-\int_{\tau=t_0}^{t} b(\tau)d\tau\} \quad \Leftrightarrow \quad x(t) = A\exp\{-\int b(t)dt\}.,$$

where $\int b(t)dt$ just means the anti-derivative of b as usual. The unknown constant A is determined with an initial condition (when one is given). The function:

$$x(t) = A\exp\{-\int b(t)dt\}$$

is called the *general solution* to $x'(t) + b(t)x(t) = 0$. The "general" refers to the fact that we have not specified A.

Example

Find the general solution to:

$$(4+t^2)\frac{dx}{dt}+2tx=0.$$

First, the coefficient on $x'(t)$ is never 0, so we can divide and put this in the form:

$$x'(t)=-\frac{2t}{4+t^2}x(t).$$

So this gives:

$$\frac{x'(t)}{x(t)}=\frac{d}{dt}(\log x(t))=-\frac{2t}{4+t^2}x(t).$$

Whence:

$$\log x(t)-\log x(0)=-\int_{\tau=0}^{t}\frac{2\tau}{4+\tau^2}d\tau=-\int_{u=4}^{4+t^2}\frac{du}{u}=-\log(4+t^2)+\log(4).$$

Thus:

$$\log x(t)=C+\log(4+t^2)$$

whence:

$$x(t)=\frac{A}{4+t^2}.$$

It's important to understand where the formula for the solution of a linear, first order, homogeneous equation comes from, which is why I basically re-derived the expression in this problem. However, it is fine to simply jump straight to:

$$x(t)=A\exp-\int\frac{2t}{4+t^2}dt=A\exp\{-\log(4+t^2)\}=\frac{A}{4+t^2}.$$

Linear Equations

Now we turn to equations of the form:

$$x'(t) + b(t)x(t) = g(t),$$

where $g(t)$ is not identically zero. Equations like this are called *linear, first order, non–homogeneous* equations. There are three steps to solve them:

(1) Find the general solution, $x_H(t)$, to $\frac{dx}{dt} + b(t)x(t) = 0$.

(2) Find a particular solution, $x_P(t)$, to $\frac{dx}{dt} + b(t)x(t) = g(t)$.

(3) The general solution then is $x(t) = x_H(t) + x_P(t)$.

The new step is the second one. Finding a solution to $x_P(t)$ can be done in a few ways. The way that "always" works is by looking for a solution of the form $x(t)x_H(t)$ where $x(t)$ is unknown. Plugging this into the equation gives:

$$g(t) = (x_H(t)x(t))' + b(t)(x_H(t)x(t)) = x(t)(x_H'(t) + b(t)x_H(t)) + x_H(t)x'(t).$$

since $x_H'(t) + b(t)x_H(t) = 0$ this gives:

$$x'(t) = \frac{g(t)}{x_H(t)} \qquad \Rightarrow \qquad x(t) = \int \frac{g(t)}{x_H(t)}dt \qquad \Rightarrow \qquad x_P(t) = x_H(t)\int \frac{g(t)}{x_H(t)}dt.$$

So, the general solution to the non–homogeneous equation is:

$$x(t) = x_H(t) + x_P(t) = A\exp\{-\int b(t)dt\} + x_H(t)\int \frac{g(t)}{x_H(t)}dt.$$

This method uses the following fact:

Theorem. Let $X_P(t)$ be any solution to $x'(t) + b(t)x(t) = g(t)$. Then the general solution to this equation can be written in the form:

$$x(t) = x_H(t) + x_P(t),$$

where $x_H(t)$ is the general solution to the corresponding homogeneous equation.

Proof. First of all, a comutation shows that any function defined as above is a solution. Indeed:

$$(x_H(t) + x_P(t))' + b(t)\left(x_H(t) + x_P(t)\right) = (x_H'(t) + b(t)x_H(t)) + (x_P'(t) + b(t)x_P(t)) = 0 + g(t).$$

Now, assume that $x(t)$ is a solution the equation. We need to show it can be written in the claimed form. That is, we need to show that:

$$x(t) = x_H(t) + x_P(t),$$

for some specific solution $x_H(t)$ to the corresponding homogeneous equation. This is the same as saying that the function $x(t) - x_P(t)$ is a solution to the corresponding homogeneous equation. And it is:

$$(x(t) - x_P(t))' + b(t)\left(x(t) - x_P(t)\right) = (x'(t) + b(t)x(t)) - (x_P'(t) + b(t)x_P(t)) = g(t) - g(t) = 0.$$

□

Example

Find the general solution to:

$$(4+t^2)\frac{dx}{dt} + 2tx = 4t.$$

First, write it as:

$$\frac{dx}{dt} + \frac{2t}{4+t^2} = \frac{4t}{4+t^2}.$$

We know that:

$$x_H(t) = \frac{A}{4+t^2}.$$

So the above says that:

$$x_P(t) = \frac{1}{4+t^2}\int \frac{4t}{4+t^2}(4+t^2)dt = \frac{2t^2}{4+t^2}.$$

So the general solution is:

$$x(t) = \frac{A}{4+t^2} + \frac{2t^2}{4+t^2}.$$

Integrating Factors

There is another way (this is the way the book teaches) to find a solution to the equation:

$$x'(t) + b(t)x(t) = g(t).$$

The idea is to multiply both sides of the equation by a to–be–determined function $\mu(t)$, so that the left hand side is the derivative of $x(t)\mu(t)$. Then we just integrate both sides and divide by μ to find x. That is, we want to find a μ such that:

$$\mu(t)x'(t) + \mu(t)b(t)x(t) = \frac{d}{dt}(\mu(t)x(t)) = \mu'(t)x(t) + \mu(t)x'(t).$$

Canceling the $\mu x'$ terms, this gives:

$$\mu(t)b(t)x(t) = \mu'(t)x(t),$$

which reduces to:

$$b(t) = \frac{\mu'(t)}{\mu(t)} = \frac{d}{dt}(\log \mu(t)).$$

Whence:

$$\log \mu(t) = \int b(t)dt \qquad \Leftrightarrow \qquad \mu(t) = \exp \int b(t)dt.$$

(Notice how similar this is to solving the corresponding homogeneous equation above!) Now we have a formula for μ, if we multiply both sides of the ODE we get:

$$\mu(t)g(t) = \mu(t)x'(t) + \mu(t)b(t)x(t) = \frac{d}{dt}(\mu(t)x(t)).$$

In other words:

$$\mu(t)x(t) = C + \int \mu(t)g(t)dt.$$

Whence:

$$x(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t)dt.$$

Comparing these two methods, observe that $\mu(t) = x_H(t)^{-1}$ (for the choice $A = 1$ in the expression for $x_H(t)$). And so this method gives the exact same expression as the method we did above. Both have their advantages.

Example

Find the solution to the IVP:

$$t\frac{dx}{dt} + 2x = 4t^2 \qquad\qquad x(1) = 2.$$

We will use integrating factors. But first we re–write it:

$$\frac{dx}{dt} + \frac{2}{t}x = 4t.$$

We want a μ so that:

$$\mu' x + \mu x' = \mu x' + \frac{2}{t}\mu x,$$

which is the same as:

$$\frac{\mu'}{\mu} = \frac{2}{t},$$

whence

$$\frac{d}{dt}\log\mu(t) = \frac{2}{t},$$

so

$$\log\mu(t) = 2\log t,$$

which gives:

$$\mu(t) = t^2.$$

So, multiplying the ODE by $\mu(t)$ gives:

$$(t^2 x(t))' = 4t^3.$$

Integrating:

$$t^2 x(t) = C + t^4,$$

solving for $x(t)$:

$$x(t) = \frac{C}{t^2} + t^2.$$

Now we determine C from the initial condition:

$$x_0 = C + 1 \qquad \Rightarrow \qquad C = x_0 - 1.$$

So the solution is:

$$x(t) = \frac{x_0 - 1}{t^2} + t^2.$$