Chapter 2 Sections 1 & 2

Chapter 2 of the book deals with first order equations of the form: There is not a practical, universal method

that can be used to solve all equations of this class. However, there are important subclasses for which there are methods. We will deal with *linear* and *separable* equations today. Later, we will learn about *exact* equations.

Linear Equations

The general form of a first order liner differential equation is:

When $g \equiv 0$, we say this is a *first order, linear, homogeneous equation.* In that case, the equation reduces to:

$$x'(t) = \qquad \Leftrightarrow \qquad = -b(t) \qquad \Leftrightarrow \qquad = -b(t)$$

In other words:

$$\log x(t) - \log x(t_0) = \qquad \Leftrightarrow \qquad x(t) = \qquad \Leftrightarrow \qquad x(t) = \qquad ,$$

The unknown constant A is determined with an initial condition (when one is given). The function:

$$x(t) = A \exp\{-\int b(t)dt\}$$

is called the *general solution* to $x'(t) + b(t)x(t) = 0$. The "general" refers to the fact that we have not specified A.

Example

Find the general solution to:

$$(4+t^2)\frac{dx}{dt}+2tx=0.$$

Linear Equations

Now we turn to equations of the form:

where $g(t)$ is not identically zero. Equations like this are called *linear, first order, non–homogeneous* equations. There are three steps to solve them:

(1) Find the general solution, $x_H(t)$, to $\frac{dx}{dt} + b(t)x(t) = 0$.

(2) Find a particular solution, $x_P(t)$, to $\frac{dx}{dt} + b(t)x(t) = g(t)$.

(3) The general solution then is $x(t) = x_H(t) + x_P(t)$.

The new step is the second one. Finding a solution to $x_P(t)$ can be done in a few ways. The way that "always" works is by looking for a solution of the form $x(t)x_H(t)$ where $x(t)$ is unknown. Plugging this into the equation gives:

since $x'_H(t) + b(t)x_H(t) = 0$ this gives:

$$x'(t) = \frac{g(t)}{x_H(t)} \qquad \Rightarrow \qquad x(t) = \int \frac{g(t)}{x_H(t)} dt \qquad \Rightarrow \qquad x_P(t) = x_H(t) \int \frac{g(t)}{x_H(t)} dt.$$

So, the general solution to the non–homogeneous equation is:

$$x(t) = x_H(t) + x_P(t) = A \exp\{-\int b(t)dt\} + x_H(t) \int \frac{g(t)}{x_H(t)} dt.$$

This method uses the following fact:

Theorem. Let $X_P(t)$ be any solution to $x'(t) + b(t)x(t) = g(t)$. Then the general solution to this equation can be written in the form:

$$x(t) = x_H(t) + x_P(t),$$

where $x_H(t)$ is the general solution to the corresponding homogeneous equation.

Proof. □

Example

Find the general solution to:

$$(4 + t^2)\frac{dx}{dt} + 2tx = 4t.$$

Integrating Factors

There is another way (this is the way the book teaches) to find a solution to the equation:

$$x'(t) + b(t)x(t) = g(t).$$

The idea is to multiply both sides of the equation by a to–be–determined function $\mu(t)$, so that the left hand side is the derivative of $x(t)\mu(t)$. Then we just integrate both sides and divide by μ to find x. That is, we want to find a μ such that:

$$\mu(t)x'(t) + \mu(t)b(t)x(t) = \frac{d}{dt}(\mu(t)x(t)) = \mu'(t)x(t) + \mu(t)x'(t).$$

Canceling the $\mu x'$ terms, this gives:

$$\mu(t)b(t)x(t) = \mu'(t)x(t),$$

which reduces to:

$$b(t) = \frac{\mu'(t)}{\mu(t)} = \frac{d}{dt}(\log \mu(t)).$$

Whence:

$$\log \mu(t) = \int b(t)dt \qquad \Leftrightarrow \qquad \mu(t) = \exp \int b(t)dt.$$

(Notice how similar this is to solving the corresponding homogeneous equation above!) Now we have a formula for μ, if we multiply both sides of the ODE we get:

$$\mu(t)g(t) = \mu(t)x'(t) + \mu(t)b(t)x(t) = \frac{d}{dt}(\mu(t)x(t)).$$

In other words:

$$\mu(t)x(t) = C + \int \mu(t)g(t)dt.$$

Whence:

$$x(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)} \int \mu(t)g(t)dt.$$

Comparing these two methods, observe that $\mu(t) = x_H(t)^{-1}$ (for the choice $A = 1$ in the expression for $x_H(t)$). And so this method gives the exact same expression as the method we did above. Both have their advantages.

Example

Find the solution to the IVP:

$$t\frac{dx}{dt} + 2x = 4t^2 \qquad\qquad x(1) = 2.$$