Chapter 2 Section 6

Consider the equation:

$$(2t + x^2) + 2tx\frac{dx}{dt} = 0.$$

This is not separable. However, the LHS does like – at least a little – like what you would get if you did chain rule on a two variable function $F(t, x)$. Indeed, if $x = x(t)$, then:

$$\frac{d}{dt}F(t,x) = \frac{\partial F}{\partial t}\frac{dt}{dt} + \frac{\partial F}{\partial x}\frac{dx}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt}.$$

If we can find such an F (we can't always – we'll see below how to determine), then the equation becomes:

$$\frac{d}{dt}F(t,x) = 0 \Longrightarrow F(t,x) = C,$$

and this defines x implicitly as a function of t in a way similar to what we saw when we did separable equations. In other words, if there is such an F, then:

$$(2t + x^2) = \frac{\partial F}{\partial t} \qquad\qquad 2tx = \frac{\partial F}{\partial x}.$$

Remember: if $F(t, x)$ is "nice" then $F_{tx} = F_{xt}$. So, if there is such an F then the functions $(2t + x^2)$ and $2tx$ must satisfy:

$$\frac{\partial}{\partial x}(2t + x^2) = \frac{\partial}{\partial t}2tx,$$

and this is the case:

$$\frac{\partial}{\partial x}(2t + x^2) = 2x = \frac{\partial}{\partial t}2tx.$$

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If $\frac{\partial F}{\partial t} = 2t + x^2$, this means that:

$$F(t,x) = \int (2t + x^2)dt + h(x) = t^2 + tx^2 + h(x).$$

Notice that the "constant of integration" is a function of x. This is because $\frac{\partial}{\partial t}h(x) = 0$.
This almost determiens F, we need to determine h. We do this by recognizing that:

$$2tx = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(t^2 + tx^2 + h(x)) = 2tx + h'(x).$$

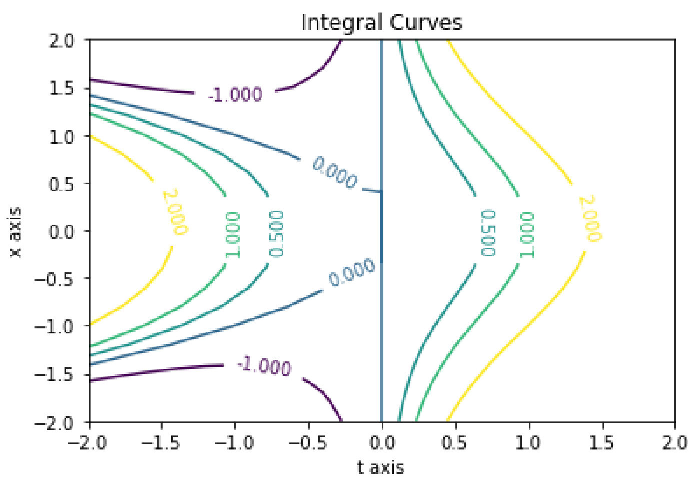
Thus:

$$h'(x) = 0 \Longrightarrow h(x) = C.$$

So we have:

$$C = F(t,x) = t^2 + tx^2.$$

If we had an initial value, we could solve for C.



Here are some curves. For the same reasons as discussed with separable equations, portions these curves will define x implicitly as a function of t when the tangent line is not vertical; this happens when $\nabla F(t,x)$ is not horizontal; which happens when F_x is not zero; which happens when $2tx$ is not zero. Thus, portions of the curve that do not contain a point (t,x) that makes $2tx = 0$ (in this case, that's $t = 0$ or $x = 0$) define x implicitly as function of t.

Solving for x is pretty easy:

$$x(t) = \pm\sqrt{\frac{C - t^2}{t}} = \pm\sqrt{\frac{C}{t} - t}.$$

If $C > 0$ for positive values of t, this is defined only when

$$\frac{C}{t} - t \geq 0 \Longrightarrow C - t^2 \geq 0 \Longrightarrow t^2 \leq C \Longrightarrow t \leq \sqrt{C}.$$

If $C > 0$, when $t < 0$, this is defined only when:

$$\frac{C}{t} - t \geq 0 \Longrightarrow C - t^2 \leq 0 \Longrightarrow t^2 \geq C \Longrightarrow t \leq -\sqrt{C}.$$

For $C < 0$, similar reasoning shows that this is not defined for any positive value of t and it is defined for all negative value of t.

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We want to study equations of the form:

$$M(t,x) + N(t,x)\frac{dx}{dt} = 0.$$

And we want to find a function $F(t,x)$ such that:

$$\frac{d}{dt}F(t,x) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} = M(t,x) + N(t,x)\frac{dx}{dt}.$$

If such an F can be found, then we know that:

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}.$$

And actually, from Calc 3, we know that if M and N satisfy this equation, and their partial derivatives are continuous, then there is such an F.

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Solve the ODE

$$(x\cos t + 2te^x) + (\sin t + t^2e^x - 1)\frac{dx}{dt} = 0.$$

We want to find a function $F(t,x)$ that satisfies:

$$\frac{d}{dt}F(t,x) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} = (x\cos t + 2te^x) + (\sin t + t^2e^x - 1)\frac{dx}{dt}.$$

Such an F exists if and only if:

$$\frac{\partial}{\partial x}(x\cos t + 2te^x) = \frac{\partial}{\partial t}(\sin t + t^2e^x - 1),$$

and this is indeed the case:

$$\frac{\partial}{\partial x}(x\cos t + 2te^x) = \cos t + 2te^x \qquad \frac{\partial}{\partial t}(\sin t + t^2e^x - 1) = \cos t + 2te^x.$$

So then:

$$F(t,x) = \int(\sin t + t^2e^x - 1)dx + h(t) = x\sin t + t^2e^x - x + h(t).$$

To find $h(t)$:

$$x\cos t + 2te^x = \frac{\partial F}{\partial t} = x\cos t + 2te^x + h'(t),$$

whence

$$h'(t) = 0 \Longrightarrow h(t) = C.$$

So, the general solution is:

$$C = F(t,x) = x\sin t + t^2e^x - x.$$

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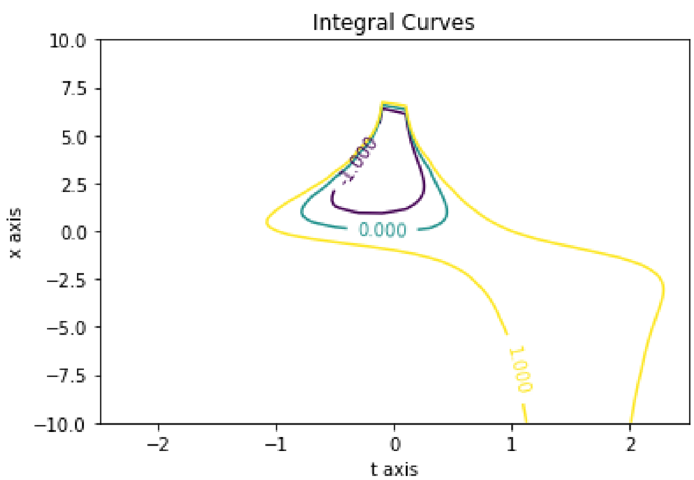

Note that the curve fail the VLT when

$$\sin t + t^2 e^x - 1 = 0$$

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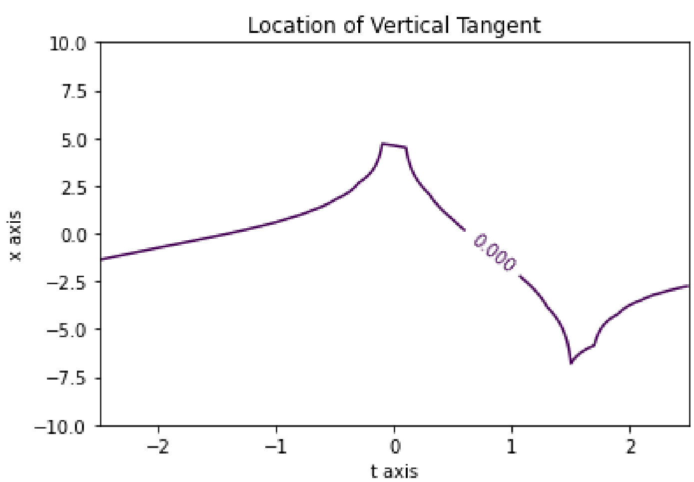
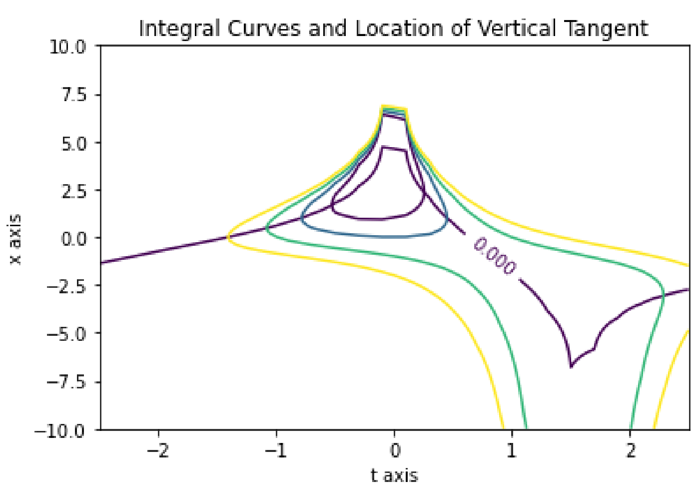


If a solution curve passes through this curve, the solution curve will have a vertical tangent at that point.



Here they are plotted on the same plot.

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Solve the ODE

$$(3tx + x^2) + (t^2 + tx)\frac{dx}{dt} = 0.$$

First, note that

$$\frac{\partial}{\partial x}(3tx + x^2) = 3t + 2x, \qquad \frac{\partial}{\partial t}(t^2 + tx) = 2t + x$$

since these aren't equal, this is not exact. But, can we transform this to an equation that is exact? Can we multiply this equation by an integrating factor, $\mu(t)$ or $\mu(x)$ that makes it exact?
If there is a $\mu = \mu(t)$, that makes it exact, this means that:

$$\frac{\partial}{\partial x}(\mu(t)(3tx + x^2)) = \frac{\partial}{\partial t}(\mu(t)(t^2 + tx)),$$

computing these gives:

$$(\frac{\partial \mu}{\partial x})(3tx + x^2) + \mu(t)\frac{\partial}{\partial x}(3tx + x^2) = (\frac{\partial \mu}{\partial t})(t^2 + tx) + \mu(t)\frac{\partial}{\partial t}(t^2 + tx).$$

Since we are assuming $\mu = \mu(t)$, this becomes:

$$\mu(t)\frac{\partial}{\partial x}(3tx + x^2) = \mu'(t)(t^2 + tx) + \mu(t)\frac{\partial}{\partial t}(t^2 + tx).$$

Re-arranging:

$$\mu'(t)(t^2 + tx) = \mu(t)\frac{\partial}{\partial x}(3tx + x^2) - \mu(t)\frac{\partial}{\partial t}(t^2 + tx) = \mu(t)\left(\frac{\partial}{\partial x}(3tx + x^2) - \frac{\partial}{\partial t}(t^2 + tx)\right),$$

whence

$$\frac{\mu'(t)}{\mu(t)} = \frac{\frac{\partial}{\partial x}(3tx + x^2) - \frac{\partial}{\partial t}(t^2 + tx)}{t^2 + tx} = \frac{(3t + 2x) - (2t + x)}{t(t + x)} = \frac{t + x}{t(t + x)} = \frac{1}{t}.$$

Thus $\mu(t) = t$. Multiplying the equation by this gives:

$$(3t^2x + tx^2) + (t^3 + t^2x)\frac{dx}{dt} = 0.$$

Then:

$$F(t, x) = \int (3t^2x + tx^2)dt + h(x) = t^3x + \frac{1}{2}t^2x^2 + h(x)$$

and

$$t^3 + t^2x = t^3 + t^2x + h'(x)$$

whence $h(x) = C$. SO the solution is:

$$C = F(t, x) = t^3x + \frac{1}{2}t^2x^2.$$

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Find an integrating factor to solve:

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0.$$

This is not exact. But, can we multiply this by a function $\mu(x)$ that will make it exact? What would such an equation need to satisfy:

$$\begin{aligned}\frac{\partial}{\partial y}(\mu(x)M(x,y)) = \frac{\partial}{\partial x}(\mu(x)N(x,y)) &\Longrightarrow \mu(x)M_y(x,y) = \mu'(x)N(x,y) + \mu(x)N_x(x,y)\\ &\Longrightarrow \mu'(x)N(x,y) = \mu(x)(M_y(x,y) - N_x(x,y))\\ &\Longrightarrow \frac{\mu'(x)}{\mu(x)} = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}.\end{aligned}$$

The only way this last line makes sense is if $(M_y - N_x)/N$ is a function of x alone. If not, then this makes no sense and there isn't an IF that is a function of x alone (in which case this formula isn't valid since we used the fact that $\mu_y = 0$).
For our particular problem:

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{(3x^2 + 2x + 3y^2) - (2x)}{x^2 + y^2} = 3.$$

Thus:

$$\frac{\mu'(x)}{\mu(x)} = 3,$$

whence $\mu(x) = e^{3x}$. Multiplying by both sides gets:

$$e^{3x}(3x^2y + 2xy + y^3) + e^{3x}(x^2 + y^2) = 0,$$

and this is exact:

$$F(x,y) = \int e^{3x}(x^2 + y^2)dy + h(x) = e^{3x}(x^2y + \frac{1}{3}y^3) + h(x).$$

To find h:

$$e^{3x}(3x^2y + 2xy + y^3) = e^{3x}(2xy) + 3e^{3x}(x^2y + \frac{1}{3}y^3) + h'(x)$$

so:

$$h'(x) = 0 \Longrightarrow h(x) = C.$$

So the solution is:

$$e^{3x}(3x^2y + y^3) = C.$$

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